Topological realization of certain resolution of the singular cohomology of BS^3

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- \square Denote by $H^*(X)$ the singular cohomology $H^*(X; \mathbb{F}_2)$.
- \square A priori, $H^*(X)$ is a \mathbb{F}_2 -graded vector space.
- As singular cohomology is a contravariant functor, the diagonal map $\Delta: X \to X \times X$ induces a product on $H^*(X)$ making it a graded commutative \mathbb{F}_2 -algebra.

 \square For every topological space X, the singular cohomology $H^*(X)$ is a module over the Steenrod algebra.

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- □ Such a module satisfies the following two conditions for all $x \in H^n(X)$:

Cartan's formula: $Sq^nx = x^2$ Instability condition: $Sq^kx = 0$ for k > n

UNSTABLE ALGEBRAS

 \square An \mathbb{F}_2 -graded commutative algebra that is also an \mathscr{A} -module satisfying Cartan's formula and the Instability condition is called an **unstable algebra**.

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- \square Denote by $\mathscr U$ the category of unstable modules.

Unstable Adams spectral sequence

$$E_2^{s,t} = \operatorname{Ext}_{\mathcal{K}}^s \left(H^* (Y), \Sigma^t H^* (X) \right) \Longrightarrow \pi_{t-s} \left(\operatorname{Map}_* \left(X, \widehat{Y}_2 \right) \right)$$

UNSTABLE ADAMS SPECTRAL SEQUENCE

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$$\operatorname{Hom}_{\mathscr{K}}(L,\Sigma K) \cong \operatorname{Hom}_{\mathscr{U}}\left(\bar{L}/\bar{L}^{2},\Sigma K\right)$$

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 $\operatorname{Hom}_{\mathscr{Y}}(M,\Sigma N) \cong \operatorname{Hom}_{\mathscr{Y}}(\Omega M,N)$

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$$\mathrm{E}_{2}^{r,s-r} = \mathrm{Ext}_{\mathscr{U}}^{r} \left(\mathrm{L}_{s-r}^{\mathrm{G}} \Omega \mathrm{QH}^{*} \left(\mathrm{Y} \right), \Sigma^{t-1} \tilde{\mathrm{H}}^{*} \left(\mathrm{X} \right) \right)$$

SULLIVAN'S CONJECTURE

$$\operatorname{Map}_*\left(\operatorname{BG}, X\right) \simeq \left\{\operatorname{point}\right\}$$

UNSTABLE ADAMS SPECTRAL SEQUENCE

$$E_2^{s,t} = \operatorname{Ext}_{\mathcal{K}}^s \left(H^* \left(S^n \right), \Sigma^t H^* \left(X \right) \right) \Longrightarrow \pi_{t-s} \left(\operatorname{Map}_* \left(X, \widehat{S^n}_2 \right) \right)$$

 $E_2^{r,s-r} = Ext_{\mathscr{U}}^r \left(L_{s-r}^G \Omega Q H^* (S^n), \Sigma^{t-1} \tilde{H}^* (X) \right)$

UNSTABLE ADAMS SPECTRAL SEQUENCE

$$\mathbf{E}_{2}^{r,s-r} = \mathbf{Ext}_{\mathcal{U}}^{r} \left(\mathbf{L}_{s-r}^{G} \Omega \mathbf{Q} \mathbf{H}^{*} (\mathbf{S}^{n}), \Sigma^{t-1} \tilde{\mathbf{H}}^{*} (\mathbf{X}) \right)$$

$$\left(\mathrm{L}_{s-r}^{\mathrm{G}}\Omega\mathrm{Q}\mathrm{H}^{*}\left(\mathrm{S}^{n}\right),\Sigma^{t-1}\tilde{\mathrm{H}}^{*}\right)$$

 $H^*(S^n) \cong U(\overline{\Sigma^n \mathbb{F}_2}) = \operatorname{Sym}^*(\overline{\Sigma^n \mathbb{F}_2}) / \langle x^2 = Sq^{|x|} x \rangle$

$$E_2^{s,t} = \operatorname{Ext}_{\mathcal{K}}^s \left(H^* \left(S^n \right), \Sigma^t H^* \left(X \right) \right) \Longrightarrow \pi_{t-s} \left(\operatorname{Map}_* \left(X, \widehat{S^n}_2 \right) \right)$$

Unstable Adams spectral sequence

$$\mathbf{E}_{2}^{s,t} = \mathbf{Ext}_{\mathcal{U}}^{s} \left(\Sigma^{n} \mathbb{F}_{2}, \Sigma^{t} \mathbf{H}^{*} \left(\mathbf{X} \right) \right) \Longrightarrow \pi_{t-s} \left(\mathbf{Map}_{*} \left(\mathbf{X}, \widehat{\mathbf{S}^{n}}_{2} \right) \right)$$

 \square Consider $\mathbb{C}P^{\infty} \simeq K(\mathbb{Z},2)$.

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- The exact sequence of groups

$$0 o \mathbb{Z} o \mathbb{Z} \left[rac{1}{2}
ight] o \mathbb{Z} \left(2^\infty
ight) o 0$$

induces the fiber sequence:

$$B\mathbb{Z}\left[\frac{1}{2}\right] \to B\mathbb{Z}(2^{\infty}) \longrightarrow \mathbb{C}P^{\infty}$$

$$\downarrow$$

$$K\left(\mathbb{Z}\left[\frac{1}{2}\right], 2\right) \to K\left(\mathbb{Z}(2^{\infty}), 2\right)$$

■ We have

$$\mathbb{Z}\left[\frac{1}{2}\right] \cong \operatorname{colimit}\left\{\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \cdots\right\}$$

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It follows that we have

$$B\mathbb{Z}\left[\frac{1}{2}\right] \simeq \operatorname{hocolim}\left\{S^{1} \xrightarrow{z \mapsto z^{2}} S^{1} \xrightarrow{z \mapsto z^{2}} S^{1} \xrightarrow{z \mapsto z^{2}} \cdots\right\}$$

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 \square It follows that $H^*\left(B\mathbb{Z}\left[\frac{1}{2}\right]\right)$ is trivial.

Therefore, the obvious monomorphism $\mathbb{Z}(2^{\infty}) \subset S^1$ induces an isomorphism in cohomology

$$H^*(\mathbb{C}P^{\infty}) \cong H^*(B\mathbb{Z}(2^{\infty})).$$

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In other words, there is an equivalence

$$\widehat{\mathrm{BZ}(2^{\infty})}_2 \to \widehat{\mathbb{C}\mathrm{P}^{\infty}}_2.$$

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☐ In other words, there is an equivalence

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☐ As a consequence, we obtain natural equivalences:

$$map_*\left(\mathbb{C}\mathrm{P}^\infty,\widehat{\mathrm{X}}_2
ight)
ightarrow map_*\left(\mathrm{B}\mathbb{Z}\left(2^\infty\right),\widehat{\mathrm{X}}_2
ight).$$

 \square The inclusion $S^0 \subset S^1$ induces the following map

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This map induces a short exact sequence of unstable modules:

$$0 \longrightarrow H^*\mathbb{C}P^\infty \longrightarrow H^*\mathbb{R}P^\infty \longrightarrow \Sigma H^*\mathbb{C}P^\infty \longrightarrow 0$$

SINGULAR COHOMOLOGY

$$H^*(BS^0) \cong \mathbb{F}_2[u] = H$$

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$$H^*(BS^0) \cong \mathbb{F}_2[u] = H$$

 $H^* (BS^1) \cong \mathbb{F}_2 [u^2] = \Phi H$

RESOLUTION

$$0 \to \operatorname{H}^*\left(\operatorname{BS}^1\right) \to \mathbb{F}_2\left[u\right] \to \Sigma \mathbb{F}_2\left[u\right] \to \Sigma^2 \mathbb{F}_2\left[u\right] \to \cdots$$

RESOLUTION

$$0 \longrightarrow \Sigma^{t} \mathbf{H}^{*}\left(\mathbf{B}\mathbf{S}^{1}\right) \longrightarrow \Sigma^{t} \mathbb{F}_{2}\left[u\right] \longrightarrow \Sigma^{t+1} \mathbb{F}_{2}\left[u\right] \longrightarrow \cdots$$

$$0 \longrightarrow \Sigma^{t} H^{*}(BS^{1}) \longrightarrow \Sigma^{t} \mathbb{F}_{2}[u] \longrightarrow \Sigma^{t+1} \mathbb{F}_{2}[u] \longrightarrow \cdots$$

$$\downarrow \qquad \qquad \uparrow \qquad \qquad \uparrow$$

$$0 \longrightarrow 0 \longrightarrow \Sigma^{t} \mathbb{F}_{2} \longrightarrow \Sigma^{t+1} \mathbb{F}_{2} \longrightarrow \cdots$$

$$0 \longrightarrow \Sigma^{t} \mathbf{H}^{*}(\mathbf{BS}^{1}) \longrightarrow \Sigma^{t} \mathbb{F}_{2}[u] \longrightarrow \Sigma^{t+1} \mathbb{F}_{2}[u] \longrightarrow \cdots$$

$$\downarrow \qquad \qquad \uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow$$

$$0 \longrightarrow 0 \longrightarrow \Sigma^{t} \mathbb{F}_{2} \xrightarrow{0} \Sigma^{t+1} \mathbb{F}_{2} \xrightarrow{0} \cdots$$

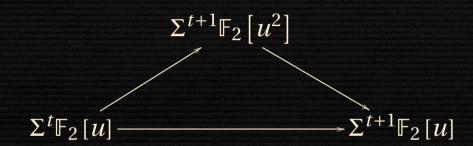
$$\operatorname{Hom}_{\mathscr{U}}\left(\Sigma^{n}\mathbb{F}_{2},\Sigma^{k}\mathbb{F}_{2}\right) \cong \operatorname{Hom}_{\mathscr{U}}\left(\Sigma^{n}\mathbb{F}_{2},\Sigma^{k}\mathbb{F}_{2}\left[u\right]\right)$$

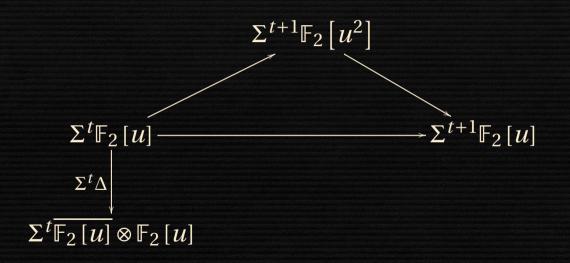
ALGEBRAIC CONNECTION

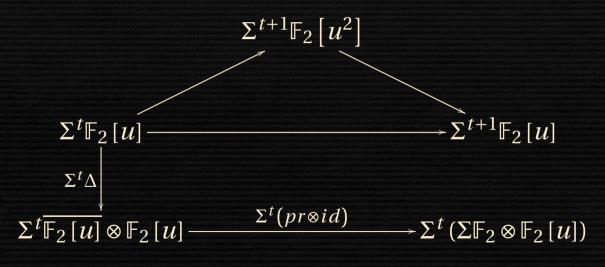
$$\operatorname{Ext}_{\mathscr{U}}^{s}\left(\Sigma^{n}\mathbb{F}_{2},\Sigma^{t}\operatorname{H}^{*}\left(\operatorname{BS}^{1}\right)\right)\cong\bigoplus_{a+b=s}\operatorname{Ext}_{\mathscr{U}}^{a}\left(\Sigma^{n}\mathbb{F}_{2},\Sigma^{t}\widetilde{\operatorname{H}}^{*}\left(\operatorname{S}^{b}\right)\right)$$

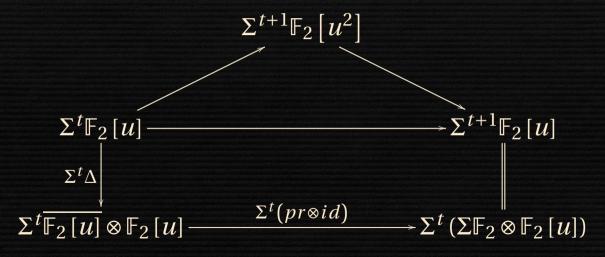
$$\Sigma^{t}\mathbb{F}_{2}[u] \cong \widetilde{H}^{*}(S^{t} \vee \Sigma^{t}\mathbb{R}P^{\infty})$$

$$\Sigma^t \mathbb{F}_2[u] \longrightarrow \Sigma^{t+1} \mathbb{F}_2[u]$$









$$S^{t+1} \vee \Sigma^{t+1} \mathbb{R} P^{\infty} \xrightarrow{\partial^t} S^t \vee \Sigma^t \mathbb{R} P^{\infty}$$

$$S^{t+1} \vee \Sigma^{t+1} \mathbb{R} P^{\infty} \xrightarrow{\delta^t} S^t \vee \Sigma^t \mathbb{R} P^{\infty}$$

$$S^{t+1} \vee \Sigma^t S^1 \wedge \mathbb{R} P^{\infty}$$

$$S^{t+1} \vee \Sigma^{t+1} \mathbb{R} P^{\infty} \xrightarrow{\vartheta^{t}} S^{t} \vee \Sigma^{t} \mathbb{R} P^{\infty}$$

$$\downarrow \\ S^{t+1} \vee \Sigma^{t} S^{1} \wedge \mathbb{R} P^{\infty} \xrightarrow{id \vee \Sigma^{t} (i \wedge id)} S^{t+1} \vee \Sigma^{t} \mathbb{R} P^{\infty} \wedge \mathbb{R} P^{\infty}$$

 $\Sigma^{t}(\mathbb{R}P^{\infty} \wedge \mathbb{R}P^{\infty}) \longrightarrow \Sigma^{t}(\mathbb{R}P^{\infty} \times \mathbb{R}P^{\infty}) \xrightarrow{\Sigma^{t} \text{mult}} \Sigma^{t} \mathbb{R}P^{\infty}$

 $\Sigma^t S^1 \xrightarrow{\Sigma^t i} \Sigma^t \mathbb{R}P^{\infty}$

THEOREM

For all integers $t \ge 1$, the following resolution can be topologically realized:

$$0 \to \Sigma^t \mathrm{H}^*\left(\mathrm{BS}^1\right) \to \Sigma^t \mathbb{F}_2\left[u\right] \to \Sigma^{t+1} \mathbb{F}_2\left[u\right] \to \cdots$$

THEOREM (FRIEDLANDER-MISLIN 1986)

Let \mathscr{L} be a Lie group with finite number of connected components, then we have a weak homotopy equivalence:

$$\operatorname{Map}_*(B\mathcal{L},X) \simeq \{\operatorname{point}\}$$

$$\operatorname{Map}_*\left(\operatorname{BS}^1,\operatorname{X}\right)\simeq\left\{\operatorname{point}\right\}$$

 $\operatorname{Map}_*\left(\operatorname{BS}^3, \operatorname{X}\right) \simeq \left\{\operatorname{point}\right\}$

SINGULAR COHOMOLOGY

$$H^*\left(BS^3\right) \cong \mathbb{F}_2\left[u^4\right]$$

BROWN-GITLER MODULES

$$\operatorname{Hom}_{\mathscr{U}}(M,J(n)) \cong \operatorname{Hom}_{\mathbb{F}_2}(M^n,\mathbb{F}_2)$$

 $\operatorname{Hom}_{\mathscr{U}}(J(m) \otimes J(n), J(m+n)) \cong \operatorname{Hom}_{\mathbb{F}_2}(\mathbb{F}_2, \mathbb{F}_2)$

$$\operatorname{Hom}_{\mathscr{U}}(J(m) \otimes J(n), J(m+n)) \cong \operatorname{Hom}_{\mathbb{F}_2}(\mathbb{F}_2, \mathbb{F}_2)$$

$$J(m) \otimes J(n) \xrightarrow{\text{mult}} J(m+n)$$

$$\operatorname{Hom}_{\mathscr{U}}(J(m) \otimes J(n), J(m+n)) \cong \operatorname{Hom}_{\mathbb{F}_2}(\mathbb{F}_2, \mathbb{F}_2)$$

 $J(m) \otimes J(n) \xrightarrow{\text{mult}} J(m+n)$

$$J\left(2^k\right)^1 \cong \mathbb{F}_2 \cong \mathbb{F}_2 \langle x_k \rangle$$

n > 0

$$\operatorname{Hom}_{\mathscr{U}}(J(m) \otimes J(n), J(m+n)) \cong \operatorname{Hom}_{\mathbb{F}_2}(\mathbb{F}_2, \mathbb{F}_2)$$

 $J(m) \otimes J(n) \xrightarrow{\text{mult}} J(m+n)$

$$J(2^{k})^{1} \cong \mathbb{F}_{2} \cong \mathbb{F}_{2} \langle x_{k} \rangle$$

$$\bigoplus J(n) \cong \mathbb{F}_{2} [x_{0}, x_{1}, x_{2}, \dots, x_{n}, \dots]$$

BROWN-GITLER MODULES

$$I^{2} = \operatorname{Im} \left(I^{1} \otimes J(2) \rightarrow J(4) \right)$$

$$I^{3} = \operatorname{Im} \left(I^{2} \otimes J(2) \rightarrow J(6) \right)$$

$$\vdots$$

$$I^{n} = \operatorname{Im} \left(I^{n-1} \otimes J(2) \rightarrow J(2n) \right)$$

 $I^0 = J(0)$

 $I^1 = J(2)$

TOPOLOGICAL REALIZATION

 $\tilde{H}^*(X^0) \cong I^0 = J(0)$

$$\tilde{H}^*(X^1) \cong I^1 = J(2)$$

$$\tilde{H}^*(X^2) \cong I^2 = \operatorname{Im}(I^1 \otimes J(2) \to J(4))$$

$$\tilde{H}^*(X^3) \cong I^3 = \operatorname{Im}(I^2 \otimes J(2) \to J(6))$$

$$\vdots$$

$$\tilde{H}^*(X^n) \cong I^n = \operatorname{Im}(I^{n-1} \otimes J(2) \to J(2n))$$

$$0 \to \mathrm{H}^*\left(\mathrm{BS}^3\right) \to \mathbb{F}_2\left[u\right] \to \mathrm{I}^1 \otimes \mathbb{F}_2\left[u\right] \to \mathrm{I}^2 \otimes \mathbb{F}_2\left[u\right] \to \cdots$$

$$\mathbf{I}^{n} \otimes \mathbb{F}_{2} [u] \longrightarrow \mathbf{I}^{n+1} \otimes \mathbb{F}_{2} [u]$$

$$I^{n} \otimes \mathbb{F}_{2}[u] \longrightarrow I^{n+1} \otimes \mathbb{F}_{2}[u]$$

$$id \otimes \Delta \downarrow$$

$$I^{n} \otimes \mathbb{F}_{2}[u] \otimes \mathbb{F}_{2}[u]$$

$$\begin{array}{c|c}
I^{n} \otimes \mathbb{F}_{2}[u] & \longrightarrow I^{n+1} \otimes \mathbb{F}_{2}[u] \\
\downarrow id \otimes \Delta & \downarrow \\
I^{n} \otimes \mathbb{F}_{2}[u] \otimes \mathbb{F}_{2}[u] & \xrightarrow{id \otimes pr \otimes id} & I^{n} \otimes J(2) \otimes \mathbb{F}_{2}[u]
\end{array}$$

$$0 \longrightarrow \Sigma^{t} \mathbf{H}^{*} (\mathbf{BS}^{3}) \longrightarrow \Sigma^{t} \mathbb{F}_{2} [u] \longrightarrow \Sigma^{t} \mathbf{I}^{1} \otimes \mathbb{F}_{2} [u] \longrightarrow \cdots$$

$$\downarrow \qquad \qquad \uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow$$

$$0 \longrightarrow 0 \longrightarrow \Sigma^{t} \mathbf{I}^{0} \longrightarrow \Sigma^{t} \mathbf{I}^{1} \longrightarrow \cdots$$

$$0 \longrightarrow \Sigma^{t} H^{*}(BS^{3}) \longrightarrow \Sigma^{t} \mathbb{F}_{2}[u] \longrightarrow \Sigma^{t} I^{1} \otimes \mathbb{F}_{2}[u] \longrightarrow \cdots$$

$$\uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow$$

$$0 \longrightarrow 0 \longrightarrow \Sigma^{t} I^{0} \longrightarrow \Sigma^{t} I^{1} \longrightarrow \cdots$$

$$0 \longrightarrow \Sigma^{t} \mathbf{H}^{*} (\mathbf{BS^{3}}) \longrightarrow \Sigma^{t} \mathbb{F}_{2} [u] \longrightarrow \Sigma^{t} \mathbf{I}^{1} \otimes \mathbb{F}_{2} [u] \longrightarrow \cdots$$

$$\downarrow \qquad \qquad \uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow$$

$$0 \longrightarrow 0 \longrightarrow \Sigma^{t} \mathbf{I}^{0} \longrightarrow \Sigma^{t} \mathbf{I}^{1} \longrightarrow \cdots$$

$$\operatorname{Hom}_{\mathscr{U}}\left(\Sigma^{n}\mathbb{F}_{2},\Sigma^{t}\mathbf{I}^{m}\right)\cong\operatorname{Hom}_{\mathscr{U}}\left(\Sigma^{n}\mathbb{F}_{2},\Sigma^{t}\mathbf{I}^{m}\otimes\mathbb{F}_{2}\left[u\right]\right)$$

ALGEBRAIC CONNECTION

$$\operatorname{Ext}_{\mathscr{U}}^{s}\left(\Sigma^{n}\mathbb{F}_{2},\Sigma^{t}\operatorname{H}^{*}\left(\operatorname{BS}^{3}\right)\right)\cong\bigoplus_{a+b-s}\operatorname{Ext}_{\mathscr{U}}^{a}\left(\Sigma^{n}\mathbb{F}_{2},\Sigma^{t}\widetilde{\operatorname{H}}^{*}\left(X^{b}\right)\right)$$

Topological realization of I^n

$I^0 = J(0) \cong \tilde{H}^*(S^0)$

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$$I^1 = J(2) \cong \tilde{H}^*(\mathbb{R}P^2)$$

$$I^{0} = J(0) \cong \tilde{H}^{*}(S^{0})$$

$$I^{1} = J(2) \cong \tilde{H}^{*}(\mathbb{R}P^{2})$$

$$X^0 := S^0$$

$$X^1 := RP^2$$

☐ There are short exact sequences:

$$0 \longrightarrow \Sigma^n \mathbb{F}_2 \longrightarrow \mathrm{I}^n \longrightarrow \Sigma^2 \mathrm{I}^{n-1} \longrightarrow 0$$

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$$0 \longrightarrow \Sigma^n \mathbb{F}_2 \longrightarrow \operatorname{I}^n \longrightarrow \Sigma^2 \operatorname{I}^{n-1} \longrightarrow 0$$

■ There are cofiber sequences:

$$\Sigma X^{n-1} \longrightarrow S^n \longrightarrow X^n$$

THEOREM

The map $\Sigma X^{n-1} \xrightarrow{-\times 2} \Sigma X^{n-1}$ factors through S^n . The homotopy cofiber of the corresponding map $\Sigma X^{n-1} \to S^n$ is the space X^n .

THANK YOU