Topological realization of certain resolution of the singular cohomology of BS^3

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Hà Nội, 12 Mars 2024

Notation

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- Denote by $H^*(X)$ the singular cohomology $H^*(X;\mathbb{F}_2)$.
- A priori, $H^*(X)$ is a $\mathbb{F}_2$-graded vector space.
- As singular cohomology is a contravariant functor, the diagonal map $\Delta : X \longrightarrow X \times X$ induces a product on $H^*(X)$ making it a graded commutative $\mathbb{F}_2$-algebra.

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- For every topological space X, the singular cohomology $H^*(X)$ is a module over the Steenrod algebra.

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- For every topological space X, the singular cohomology $H^*(X)$ is a module over the Steenrod algebra.
- Such a module satisfies the following two conditions for all $x \in H^n(X)$:

$$\text{Cartan's formula:} \quad Sq^n x = x^2$$

$$\text{Instability condition:} \quad Sq^k x = 0 \text{ for } k > n$$

Unstable Algebras

- An $\mathbb{F}_2$-graded commutative algebra that is also an $\mathscr{A}$-module satisfying Cartan's formula and the Instability condition is called an **unstable algebra**.

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- An $\mathbb{N}$-graded module over the Steenrod algebra $\mathscr{A}$ satisfying the Instability condition is called an **unstable module**.
- Denote by $\mathscr{U}$ the category of unstable modules.

Unstable Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{K}}\left(H^*(Y), \Sigma^t H^*(X)\right) \Longrightarrow \pi_{t-s}\left(\mathrm{Map}_*\left(X, \widehat{Y}_2\right)\right)$$

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$$\mathrm{Hom}_{\mathcal{K}}(L, \Sigma K) \cong \mathrm{Hom}_{\mathcal{U}}\left(\bar{L}/\bar{L}^2, \Sigma K\right)$$

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$$\mathrm{Hom}_{\mathcal{U}}(M, \Sigma N) \cong \mathrm{Hom}_{\mathcal{U}}(\Omega M, N)$$

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$$\Uparrow$$

$$E_2^{r,s-r} = \mathrm{Ext}^r_{\mathcal{U}}\left(L^G_{s-r}\Omega Q H^*(Y), \Sigma^{t-1}\tilde{H}^*(X)\right)$$

Sullivan's conjecture

$$\boxed{\mathrm{Map}_*\left(BG\ ,X\right) \simeq \{\text{point}\}}$$

Unstable Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{K}}\left(H^*(S^n), \Sigma^t H^*(X)\right) \Longrightarrow \pi_{t-s}\left(\mathrm{Map}_*(X, \widehat{S^n}_2)\right)$$

$$\Uparrow$$

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$$E_2^{r,s-r} = \mathrm{Ext}^r_{\mathcal{U}}\left(L^G_{s-r}\Omega QH^*(S^n), \Sigma^{t-1}\tilde{H}^*(X)\right)$$

$$H^*(S^n) \cong U(\Sigma^n \mathbb{F}_2) = \mathrm{Sym}^*(\Sigma^n \mathbb{F}_2) \big/ \langle x^2 = Sq^{|x|}x \rangle$$

Unstable Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathscr{U}}\left(\Sigma^n \mathbb{F}_2, \Sigma^t H^*(X)\right) \Longrightarrow \pi_{t-s}\left(\mathrm{Map}_*\left(X, \widehat{S^n}_2\right)\right)$$

- Consider $\mathbb{C}P^{\infty} \simeq K(\mathbb{Z}, 2)$.

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- The exact sequence of groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}\left[\frac{1}{2}\right] \longrightarrow \mathbb{Z}\left(2^\infty\right) \longrightarrow 0$$

induces the fiber sequence:

$$\begin{array}{ccccc} B\mathbb{Z}\left[\tfrac{1}{2}\right] \to B\mathbb{Z}\left(2^\infty\right) & \longrightarrow & \mathbb{C}P^\infty & & \\ & & \downarrow & & \\ & & K\left(\mathbb{Z}\left[\tfrac{1}{2}\right],2\right) & \to & K\left(\mathbb{Z}\left(2^\infty\right),2\right) \end{array}$$

- We have

$$\mathbb{Z}\left[\frac{1}{2}\right] \cong \operatorname{colimit}\left\{\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \cdots\right\}$$

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- It follows that we have

$$\mathrm{B}\mathbb{Z}\left[\frac{1}{2}\right] \simeq \operatorname{hocolim}\left\{S^1 \xrightarrow{z \mapsto z^2} S^1 \xrightarrow{z \mapsto z^2} S^1 \xrightarrow{z \mapsto z^2} \cdots\right\}$$

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- It follows that $\mathrm{H}^*\left(\mathrm{B}\mathbb{Z}\left[\frac{1}{2}\right]\right)$ is trivial.

- Therefore, the obvious monomorphism $\mathbb{Z}(2^\infty) \subset S^1$ induces an isomorphism in cohomology

$$H^*\left(\mathbb{C}P^\infty\right) \cong H^*\left(B\mathbb{Z}\left(2^\infty\right)\right).$$

- Therefore, the obvious monomorphism $\mathbb{Z}(2^\infty) \subset S^1$ induces an isomorphism in cohomology

$$H^*\left(\mathbb{C}P^\infty\right) \cong H^*\left(B\mathbb{Z}\left(2^\infty\right)\right).$$

- In other words, there is an equivalence

$$\widehat{B\mathbb{Z}(2^\infty)}_2 \to \widehat{\mathbb{C}P^\infty}_2.$$

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$$H^*(\mathbb{C}P^\infty) \cong H^*(B\mathbb{Z}(2^\infty)).$$

- In other words, there is an equivalence

$$\widehat{B\mathbb{Z}(2^\infty)}_2 \to \widehat{\mathbb{C}P^\infty}_2.$$

- As a consequence, we obtain natural equivalences:

$$map_*(\mathbb{C}P^\infty, \hat{X}_2) \to map_*(B\mathbb{Z}(2^\infty), \hat{X}_2).$$

- The inclusion $S^0 \subset S^1$ induces the following map

$$BS^0 \to BS^1$$

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- This map induces a short exact sequence of unstable modules:

$$0 \to H^*\mathbb{C}P^\infty \to H^*\mathbb{R}P^\infty \to \Sigma H^*\mathbb{C}P^\infty \to 0$$

Singular cohomology

$$H^*\left(BS^0\right) \cong \mathbb{F}_2[u] = H$$

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$$H^*(BS^1) \cong \mathbb{F}_2[u^2]$$

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Resolution

$$0 \to \mathrm{H}^*\left(\mathrm{BS}^1\right) \to \mathbb{F}_2[u] \to \Sigma\mathbb{F}_2[u] \to \Sigma^2\mathbb{F}_2[u] \to \cdots$$

Resolution

$$0 \to \Sigma^t \mathrm{H}^*\left(\mathrm{BS}^1\right) \to \Sigma^t \mathbb{F}_2[u] \to \Sigma^{t+1}\mathbb{F}_2[u] \to \cdots$$

Resolution

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^t \mathrm{H}^*\left(\mathrm{BS}^1\right) & \longrightarrow & \Sigma^t \mathbb{F}_2[u] & \longrightarrow & \Sigma^{t+1} \mathbb{F}_2[u] & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \Sigma^t \mathbb{F}_2 & \longrightarrow & \Sigma^{t+1} \mathbb{F}_2 & \longrightarrow & \cdots
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\end{array}$$

RESOLUTION

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^t H^*(BS^1) & \longrightarrow & \Sigma^t \mathbb{F}_2[u] & \longrightarrow & \Sigma^{t+1}\mathbb{F}_2[u] & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \Sigma^t \mathbb{F}_2 & \xrightarrow{0} & \Sigma^{t+1}\mathbb{F}_2 & \xrightarrow{0} & \cdots
\end{array}$$

$$\mathrm{Hom}_{\mathscr{U}}\left(\Sigma^n \mathbb{F}_2, \Sigma^k \mathbb{F}_2\right) \cong \mathrm{Hom}_{\mathscr{U}}\left(\Sigma^n \mathbb{F}_2, \Sigma^k \mathbb{F}_2[u]\right)$$

Algebraic connection

$$\mathrm{Ext}^{s}_{\mathscr{U}}\left(\Sigma^{n}\mathbb{F}_2, \Sigma^{t}\mathrm{H}^{*}\left(\mathrm{BS}^{1}\right)\right) \cong \bigoplus_{a+b=s} \mathrm{Ext}^{a}_{\mathscr{U}}\left(\Sigma^{n}\mathbb{F}_2, \Sigma^{t}\tilde{\mathrm{H}}^{*}\left(\mathrm{S}^{b}\right)\right)$$

Remark

$$\Sigma^t \mathbb{F}_2[u] \cong \tilde{\mathrm{H}}^*\left(\mathrm{S}^t \vee \Sigma^t \mathbb{R}\mathrm{P}^\infty\right)$$

Remark

$$\Sigma^t \mathbb{F}_2[u] \longrightarrow \Sigma^{t+1} \mathbb{F}_2[u]$$

Remark

$$\begin{array}{ccccc} & & \Sigma^{t+1}\mathbb{F}_2[u^2] & & \\ & \nearrow & & \searrow & \\ \Sigma^{t}\mathbb{F}_2[u] & & \longrightarrow & & \Sigma^{t+1}\mathbb{F}_2[u] \end{array}$$

Remark

$$\begin{array}{ccc} & \Sigma^{t+1}\mathbb{F}_2[u^2] & \\ \nearrow & & \searrow \\ \Sigma^{t}\mathbb{F}_2[u] & \longrightarrow & \Sigma^{t+1}\mathbb{F}_2[u] \\ \Big\downarrow{\scriptstyle \Sigma^{t}\Delta} & & \\ \Sigma^{t}\overline{\mathbb{F}_2[u]}\otimes\mathbb{F}_2[u] & & \end{array}$$

REMARK

$$
\begin{array}{ccccc}
 & & \Sigma^{t+1}\mathbb{F}_2[u^2] & & \\
 & \nearrow & & \searrow & \\
\Sigma^t\mathbb{F}_2[u] & & \longrightarrow & & \Sigma^{t+1}\mathbb{F}_2[u] \\
\Big\downarrow{\scriptstyle \Sigma^t\Delta} & & & & \\
\Sigma^t\overline{\mathbb{F}_2[u]}\otimes\mathbb{F}_2[u] & & \xrightarrow{\Sigma^t(pr\otimes id)} & & \Sigma^t(\Sigma\mathbb{F}_2\otimes\mathbb{F}_2[u])
\end{array}
$$

Remark

$$\begin{array}{ccc} & \Sigma^{t+1}\mathbb{F}_2[u^2] & \\ \nearrow & & \searrow \\ \Sigma^{t}\mathbb{F}_2[u] & \longrightarrow & \Sigma^{t+1}\mathbb{F}_2[u] \\ \Big\downarrow {\scriptstyle \Sigma^{t}\Delta} & & \Big\| \\ \Sigma^{t}\overline{\mathbb{F}_2[u]}\otimes\mathbb{F}_2[u] & \xrightarrow{\Sigma^{t}(pr\otimes id)} & \Sigma^{t}(\Sigma\mathbb{F}_2\otimes\mathbb{F}_2[u]) \end{array}$$

Remark for $t \geq 1$

$$S^{t+1} \vee \Sigma^{t+1}\mathbb{R}P^{\infty} \xrightarrow{\partial^t} S^{t} \vee \Sigma^{t}\mathbb{R}P^{\infty}$$

Remark for $t \geq 1$

$$\begin{array}{ccc} S^{t+1} \vee \Sigma^{t+1}\mathbb{R}P^{\infty} & \xrightarrow{\partial^t} & S^t \vee \Sigma^t \mathbb{R}P^{\infty} \\ \| & & \\ S^{t+1} \vee \Sigma^t S^1 \wedge \mathbb{R}P^{\infty} & & \end{array}$$

Remark for $t \geq 1$

$$\begin{array}{ccc} S^{t+1} \vee \Sigma^{t+1}\mathbb{R}P^\infty & \xrightarrow{\partial^t} & S^t \vee \Sigma^t\mathbb{R}P^\infty \\ \| & & \\ S^{t+1} \vee \Sigma^t S^1 \wedge \mathbb{R}P^\infty & \xrightarrow{id \vee \Sigma^t(i \wedge id)} & S^{t+1} \vee \Sigma^t\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \end{array}$$

Remark for $t \geq 1$

$$\begin{array}{ccc} S^{t+1}\vee\Sigma^{t+1}\mathbb{RP}^{\infty} & \xrightarrow{\partial^{t}} & S^{t}\vee\Sigma^{t}\mathbb{RP}^{\infty} \\ \Vert & & \uparrow{\scriptstyle\varphi} \\ S^{t+1}\vee\Sigma^{t}S^{1}\wedge\mathbb{RP}^{\infty} & \xrightarrow{id\vee\Sigma^{t}(i\wedge id)} & S^{t+1}\vee\Sigma^{t}\mathbb{RP}^{\infty}\wedge\mathbb{RP}^{\infty} \end{array}$$

REMARK FOR $t \geq 1$

$$\begin{array}{ccc}
S^{t+1} \vee \Sigma^{t+1}\mathbb{R}P^{\infty} & \xrightarrow{\partial^t} & S^{t} \vee \Sigma^{t}\mathbb{R}P^{\infty} \\
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\end{array}$$

$$S^{t+1} = \!=\!=\!= \Sigma^{t} S^{1} \xrightarrow{\Sigma^{t} i} \Sigma^{t}\mathbb{R}P^{\infty}$$

Remark for $t \geq 1$

$$\begin{array}{ccc}
S^{t+1} \vee \Sigma^{t+1}\mathbb{R}P^{\infty} & \xrightarrow{\partial^t} & S^{t} \vee \Sigma^{t}\mathbb{R}P^{\infty} \\
\Vert & & \uparrow \varphi \\
S^{t+1} \vee \Sigma^{t} S^{1} \wedge \mathbb{R}P^{\infty} & \xrightarrow{id \vee \Sigma^{t}(i \wedge id)} & S^{t+1} \vee \Sigma^{t}\mathbb{R}P^{\infty} \wedge \mathbb{R}P^{\infty}
\end{array}$$

$$S^{t+1} = \Sigma^{t} S^{1} \xrightarrow{\Sigma^{t} i} \Sigma^{t}\mathbb{R}P^{\infty}$$

$$\Sigma^{t}(\mathbb{R}P^{\infty} \wedge \mathbb{R}P^{\infty}) \longrightarrow \Sigma^{t}(\mathbb{R}P^{\infty} \times \mathbb{R}P^{\infty}) \xrightarrow{\Sigma^{t}\mathrm{mult}} \Sigma^{t}\mathbb{R}P^{\infty}$$

Theorem

For all integers $t \geq 1$, the following resolution can be topologically realized:

$$0 \to \Sigma^t H^*(BS^1) \to \Sigma^t \mathbb{F}_2[u] \to \Sigma^{t+1} \mathbb{F}_2[u] \to \cdots$$

Theorem (Friedlander-Mislin 1986)

Let $\mathscr{L}$ be a Lie group with finite number of connected components, then we have a weak homotopy equivalence:

$$\boxed{\mathrm{Map}_*(B\mathscr{L}, X) \simeq \{\text{point}\}}$$

Extending to Lie groups with finite number of connected components

$$\boxed{\mathrm{Map}_*\left(BS^1, X\right) \simeq \{\text{point}\}}$$

Extending to Lie groups with finite number of connected components

$$\boxed{\mathrm{Map}_*\left(\mathrm{BS}^3, \mathrm{X}\right) \simeq \{\mathrm{point}\}}$$

Singular cohomology

$$\mathrm{H}^*\left(\mathrm{BS}^3\right) \cong \mathbb{F}_2\left[u^4\right]$$

Brown-Gitler modules

$$\mathrm{Hom}_{\mathscr{U}}\left(M, J(n)\right) \cong \mathrm{Hom}_{\mathbb{F}_2}\left(M^n, \mathbb{F}_2\right)$$

Miller algebra

$$\mathrm{Hom}_{\mathscr{U}}\left(\mathrm{J}\left(m\right)\otimes\mathrm{J}\left(n\right),\mathrm{J}\left(m+n\right)\right)\cong\mathrm{Hom}_{\mathbb{F}_2}\left(\mathbb{F}_2,\mathbb{F}_2\right)$$

Miller algebra

$$\mathrm{Hom}_{\mathscr{U}}(\mathrm{J}(m)\otimes \mathrm{J}(n), \mathrm{J}(m+n)) \cong \mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_2, \mathbb{F}_2)$$

$$\mathrm{J}(m)\otimes \mathrm{J}(n) \xrightarrow{\text{mult}} \mathrm{J}(m+n)$$

Miller algebra

$$\mathrm{Hom}_{\mathscr{U}}(\mathrm{J}(m) \otimes \mathrm{J}(n), \mathrm{J}(m+n)) \cong \mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_2, \mathbb{F}_2)$$

$$\mathrm{J}(m) \otimes \mathrm{J}(n) \xrightarrow{\text{mult}} \mathrm{J}(m+n)$$

$$\mathrm{J}\left(2^k\right)^1 \cong \mathbb{F}_2 \cong \mathbb{F}_2\langle x_k \rangle$$

Miller algebra

$$\mathrm{Hom}_{\mathscr{U}}\left(\mathrm{J}(m) \otimes \mathrm{J}(n), \mathrm{J}(m+n)\right) \cong \mathrm{Hom}_{\mathbb{F}_2}\left(\mathbb{F}_2, \mathbb{F}_2\right)$$

$$\mathrm{J}(m) \otimes \mathrm{J}(n) \xrightarrow{\text{mult}} \mathrm{J}(m+n)$$

$$\mathrm{J}\left(2^k\right)^1 \cong \mathbb{F}_2 \cong \mathbb{F}_2 \langle x_k \rangle$$

$$\bigoplus_{n \geq 0} \mathrm{J}(n) \cong \mathbb{F}_2 [x_0, x_1, x_2, \ldots, x_n, \ldots]$$

Brown-Gitler modules

$$\begin{aligned}
I^0 &= J(0) \\
I^1 &= J(2) \\
I^2 &= \mathrm{Im}\left(I^1 \otimes J(2) \to J(4)\right) \\
I^3 &= \mathrm{Im}\left(I^2 \otimes J(2) \to J(6)\right) \\
&\vdots \\
I^n &= \mathrm{Im}\left(I^{n-1} \otimes J(2) \to J(2n)\right)
\end{aligned}$$

Topological Realization

$$\tilde{H}^*(X^0) \cong I^0 = J(0)$$

$$\tilde{H}^*(X^1) \cong I^1 = J(2)$$

$$\tilde{H}^*(X^2) \cong I^2 = \mathrm{Im}\left(I^1 \otimes J(2) \to J(4)\right)$$

$$\tilde{H}^*(X^3) \cong I^3 = \mathrm{Im}\left(I^2 \otimes J(2) \to J(6)\right)$$

$$\vdots$$

$$\tilde{H}^*(X^n) \cong I^n = \mathrm{Im}\left(I^{n-1} \otimes J(2) \to J(2n)\right)$$

Resolution

$$0 \to H^*(BS^3) \to \mathbb{F}_2[u] \to I^1 \otimes \mathbb{F}_2[u] \to I^2 \otimes \mathbb{F}_2[u] \to \cdots$$

Differentials

$$\mathrm{I}^n \otimes \mathbb{F}_2[u] \longrightarrow \mathrm{I}^{n+1} \otimes \mathbb{F}_2[u]$$

Differentials

$$\begin{array}{ccc} I^n \otimes \mathbb{F}_2[u] & \longrightarrow & I^{n+1} \otimes \mathbb{F}_2[u] \\ \Big\downarrow {\scriptstyle id \otimes \Delta} & & \\ I^n \otimes \mathbb{F}_2[u] \otimes \mathbb{F}_2[u] & & \end{array}$$

Differentials

$$
\begin{array}{ccc}
\mathrm{I}^{n} \otimes \mathbb{F}_2[u] & \longrightarrow & \mathrm{I}^{n+1} \otimes \mathbb{F}_2[u] \\
{\scriptstyle id\otimes\Delta}\big\downarrow & & \\
\mathrm{I}^{n} \otimes \mathbb{F}_2[u] \otimes \mathbb{F}_2[u] & \xrightarrow{\;id\otimes pr\otimes id\;} & \mathrm{I}^{n} \otimes \mathrm{J}(2) \otimes \mathbb{F}_2[u]
\end{array}
$$

Differentials

$$
\begin{array}{ccc}
I^{n} \otimes \mathbb{F}_2[u] & \longrightarrow & I^{n+1} \otimes \mathbb{F}_2[u] \\
\Big\downarrow {\scriptstyle id \otimes \Delta} & & \Big\uparrow {\scriptstyle \mathrm{mult} \otimes id} \\
I^{n} \otimes \mathbb{F}_2[u] \otimes \mathbb{F}_2[u] & \xrightarrow{id \otimes pr \otimes id} & I^{n} \otimes J(2) \otimes \mathbb{F}_2[u]
\end{array}
$$

Resolution

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^t \mathrm{H}^*\left(\mathrm{BS}^3\right) & \longrightarrow & \Sigma^t \mathbb{F}_2[u] & \longrightarrow & \Sigma^t \mathrm{I}^1 \otimes \mathbb{F}_2[u] & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \Sigma^t \mathrm{I}^0 & \longrightarrow & \Sigma^t \mathrm{I}^1 & \longrightarrow & \cdots
\end{array}$$

Resolution

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\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^t \mathrm{H}^*\left(\mathrm{BS}^3\right) & \longrightarrow & \Sigma^t \mathbb{F}_2[u] & \longrightarrow & \Sigma^t \mathrm{I}^1 \otimes \mathbb{F}_2[u] & \longrightarrow & \cdots \\
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0 & \longrightarrow & 0 & \longrightarrow & \Sigma^t \mathrm{I}^0 & \xrightarrow{0} & \Sigma^t \mathrm{I}^1 & \xrightarrow{0} & \cdots
\end{array}
$$

Resolution

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^t H^*\left(BS^3\right) & \longrightarrow & \Sigma^t \mathbb{F}_2[u] & \longrightarrow & \Sigma^t I^1 \otimes \mathbb{F}_2[u] & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \Sigma^t I^0 & \xrightarrow{0} & \Sigma^t I^1 & \xrightarrow{0} & \cdots
\end{array}$$

$$\mathrm{Hom}_{\mathscr{U}}\left(\Sigma^n \mathbb{F}_2, \Sigma^t I^m\right) \cong \mathrm{Hom}_{\mathscr{U}}\left(\Sigma^n \mathbb{F}_2, \Sigma^t I^m \otimes \mathbb{F}_2[u]\right)$$

Algebraic connection

$$\mathrm{Ext}^s_{\mathscr{U}}\left(\Sigma^n\mathbb{F}_2, \Sigma^t \mathrm{H}^*\left(\mathrm{BS}^3\right)\right) \cong \bigoplus_{a+b=s} \mathrm{Ext}^a_{\mathscr{U}}\left(\Sigma^n\mathbb{F}_2, \Sigma^t \tilde{\mathrm{H}}^*\left(\mathrm{X}^b\right)\right)$$

Topological realization of I^n

$$I^0 = J(0) \cong \tilde{H}^*\left(S^0\right)$$

$$I^0 = J(0) \cong \tilde{H}^*\left(S^0\right)$$

$$I^1 = J(2) \cong \tilde{H}^*\left(\mathbb{R}P^2\right)$$

$$I^0 = J(0) \cong \tilde{H}^*(S^0)$$

$$I^1 = J(2) \cong \tilde{H}^*(\mathbb{R}P^2)$$

$$X^0 := S^0$$

$$X^1 := RP^2$$

Remark

- There are short exact sequences:

$$0 \longrightarrow \Sigma^n \mathbb{F}_2 \longrightarrow \mathrm{I}^n \longrightarrow \Sigma^2 \mathrm{I}^{n-1} \longrightarrow 0$$

Remark

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- There are cofiber sequences:

$$\Sigma X^{n-1} \longrightarrow S^n \longrightarrow X^n$$

Theorem

The map $\Sigma X^{n-1} \xrightarrow{-\times 2} \Sigma X^{n-1}$ factors through S^n. The homotopy cofiber of the corresponding map $\Sigma X^{n-1} \to S^n$ is the space X^n.

THANK YOU