A theorem of Pitman type for the simple random walk on $\mathbb{Z}^d$

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Contents

1 Introduction

Pitman's theorem for a one-dimensional Brownian motion $B(t)$ states that $B(t) - 2\mathbf{m}(t)$ is a Bessel process, were $B(0) = 0$ and $\mathbf{m}(t)$ denotes the minimum of $B(s), 0 \leq s \leq t$. Since the Brownian motion can be obtained, after a scaling limit, from a coin-tossing random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}$, it is natural to prove a similar statement for this random walk and its extension to higher dimensional random walks.

Following [4], we prove with elementary arguments that $(S_n - 2\mathbf{m}_n)_{n\geq 0}$, with $S_0 = 0$ and $\mathbf{m}_n = \min(S_k \mid 0 \leq k \leq n)$, is a Markov chain on $\mathbb{N}$ whose transition function $p(x, y)$ can be computed. This transition probability is expressed as the Doob transform of the transitions of the simple random walk, obtained using a positive harmonic function of this random walk confined to $\mathbb{N}$. The Pitman transform therefore proposes a trajectory interpretation of this Doob transform and can be extended in higher dimensions.

In [4], H. Miyazaki & H. Tanaka extend this result in higher dimension. They claim that the approach presented here in dimension $d = 1$ may also be applied to the case $d \geq 2$ without giving any detail and writing "*the argument will be quite messy*" [4, Introduction]. They employ another method based on the simple observation that the coordinate processes of a simple random walk on $\mathbb{Z}^d, d \geq 2$, with *continuous* time are independent although this is not true for the case of discrete time. This extension is not presented here.

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1.1 Notations

Let $(\varepsilon)_{i\geq 1}$ be a sequence of i.i.d. random variables with values in $\{-1,+1\}$ and Bernoulli distribution $\mu_p = p\delta_1 + q\delta_{-1}$ with $0<p,q<1$ and $p+q=1$.

We set $S_0 = 0,\ S_n = \varepsilon_1 + \ldots + \varepsilon_n$ and $\mathbf{m}_n = \min(S_0,\ldots,S_n)$ for any $n\geq 1$.

The process $(S_n)_{n\geq 0}$ is an irreducible Markov chain on $\mathbb{Z}$ with transition matrix $P = (p(x,y))_{x,y\in\mathbb{Z}}$ given by

$$p(x,y) = \begin{cases} p & \text{if } y = x+1 \\ q & \text{if } y = x-1 \\ 0 & \text{otherwise.} \end{cases}$$

The Markov chain $(S_n)_{n\geq 0}$ is defined on the canonical probability space $(\mathbb{Z}^{\mathbb{N}}, \mathcal{P}(\mathbb{Z})^{\otimes\mathbb{N}}, (\mathbb{P}_x)_{x\in\mathbb{Z}})$ where the measure $\mathbb{P}_x$ is the unique probability measure on $\mathcal{P}(\mathbb{Z})^{\otimes\mathbb{N}}$ satisfying the property: for any $n\geq 1$ and $(x_1,\ldots,x_n)\in\mathbb{Z}^n$

$$\mathbb{P}_x(S_1 = x_1,\ldots,S_n = x_n) = \underbrace{p(x,x_1)\ldots p(x_{n-1},x_n)}_{=:p(x,x_1,\ldots,x_n)}.$$

notice that $\mathbb{P}_{x_0}(S_1 = x_1,\ldots,S_n = x_n) > 0$ if and only if $|x_i - x_{i-1}| = 1$ for $1\leq i\leq n$.

The behavior of the random walk depends on the value of the expectation $\mathbb{E}(\varepsilon_1) = p-q$.

- Centered case $\mathbb{E}(\varepsilon_1) = 0$ (or equivalently $p = 1/2$)

The r.w. is centered and recurrent on $\mathbb{Z}$ and

$$\liminf_{n\to+\infty} S_n = \lim_{n\to+\infty} \mathbf{m}_n = -\infty \quad \text{and} \quad \limsup_{n\to+\infty} S_n = +\infty.$$

- negative drift $\mathbb{E}(\varepsilon_1) < 0$ (or equivalently $p < 1/2$)

The r.w. is transient and

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \mathbf{m}_n = -\infty.$$

- Positive drift $\mathbb{E}(\varepsilon_1) > 0$ (or equivalently $p > 1/2$)

The r.w. is transient and

$$\lim_{n\to+\infty} S_n = +\infty \quad \text{and} \quad \liminf_{n\to+\infty} \mathbf{m}_n > -\infty.$$

2 Decomposition of the trajectories of the random walk $(S_n)_{n\geq 0}$

2.1 On the descending ladder times and process

Let us consider the sequence $(d_k)_{k\geq 0}$ of **descending** random times associated to the random walk $(S_n)_{n\geq 0}$ and defined by $d_0 = 0$ and, for $k\geq 1$,

$$d_k := \inf\{n > d_{k-1} \mid S_n < S_{d_{k-1}}\}$$

with the convention $\inf\emptyset = +\infty$ and $d_k = +\infty$ if $d_{k-1} = +\infty$.

The d_k are stopping times with respect to $(S_n)_{n\geq 0}$: the event $(d_k = \ell)$ depends only on the positions $S_0, S_1, \ldots, S_\ell$ for any $k,\ell\geq 0$. Furthermore,

- if $p\leq 1/2$ then $\mathbb{P}(d_k < +\infty) = 1$ for any $k\geq 0$;

- if $p > 1/2$ then $\mathbb{P}(d_1 < +\infty) < 1$, so that $\mathbb{P}(d_k < +\infty) = (\mathbb{P}(d_1 < +\infty))^k < 1$ for any $k\geq 1$.

The following statement is of major interest:

Proposition 2.1. *The sequence* $\Big((d_k - d_{k-1})\mathbf{1}_{d_{k-1}<+\infty}\Big)_{k\geq 1}$ *is iid.*

Notice at last that, for any $k \geq 0$, it holds $\quad S_{d_k}\mathbf{1}_{d_k<+\infty} = -k \quad$ and

$$\min(S_\ell \mid 0 \leq \ell < d_{k+1})\mathbf{1}_{d_k<+\infty} = -k.$$

2.2 Decomposition of the trajectories of the random walk $(S_n)_{n\geq 0}$

• By the above, when $p \leq 1/2$, all the stopping times $d_k, k \geq 0$ are $\mathbb{P}$-as finite. Hence the sequence $S_0, S_1, S_2, \ldots$ may be decomposed as

$$\underbrace{S_0 = 0, S_1, \ldots, S_{d_1-1}}_{=:\mathcal{D}_1}, \underbrace{S_{d_1} = -1, S_{d_1+1}, \ldots, S_{d_2-1}}_{=:\mathcal{D}_2}, \quad \ldots \quad , \underbrace{S_{d_k} = -k, S_{d_k+1}, \ldots, S_{d_{k+1}-1}}_{=:\mathcal{D}_k}, \quad \ldots \tag{2.1}$$

The random variables D_k are iid and concern the successive descending excursions of $(S_n)_{n\geq 0}$.

• When $p > 1/2$, it holds $\mathbb{P}(d_1 < +\infty) < 1$; consequently $\mathbb{P}(\forall k \geq 1, d_k < +\infty) = 0$. Hence, there exist a random time $\kappa \geq 0$ such that $\mathbb{P}(d_\kappa < +\infty, d_{\kappa+1} = +\infty) = 1$ so that the sequence $S_0, S_1, S_2, \ldots$ may be decomposed as

$$\underbrace{S_0, S_1, \ldots, S_{d_1-1}}_{=:\mathcal{D}_1}, \quad \ldots\ldots \quad , \underbrace{S_{d_{\kappa-1}}, \ldots, S_{d_\kappa-1}}_{=:\mathcal{D}_{\kappa-1}}, \underbrace{S_{d_\kappa} = -\kappa, S_{d_\kappa+1}, S_{d_\kappa+2}, \quad \ldots}_{=:\mathcal{D}_\infty} \tag{2.2}$$

All the $S_{d_\kappa+\ell}, \ell \geq 0$, are greater than $-\kappa$. The random variables $\mathcal{D}_1\mathbf{1}_{d_1<+\infty}, \mathcal{D}_2\mathbf{1}_{d_2<+\infty}, \ldots$ are also iid and the variable $\mathcal{D}_\infty$ has same distribution as $\mathcal{D}_1\mathbf{1}_{d_1=+\infty}$.

3 Pitman's theorem in the case $d = 1$

3.1 The killed random walk on $\mathbb{N}$ and its Doob's transform

It is natural to consider the following operator P^+ given by: for any $\varphi : \mathbb{Z} \to \mathbb{R}^+$ and $x \geq 0$,

$$P^+\varphi(x) := \mathbb{E}_x(\varphi(S_1), d_1 \geq 1) = \mathbb{E}(\varphi(x + \varepsilon_1), x + \varepsilon_1 \geq 0).$$

This operator is submarkovian since $P^+\mathbf{1}(0) = \mathbb{P}(\varepsilon_1 = 1) = p < 1$; nevertheless, by Lemma 4.1, the following function h

$$h(x) = \begin{cases} x+1 & \text{if} \quad p = 1/2, \\ \gamma^{x+1} - 1 & \text{if} \quad q > p, \\ 1 - \gamma^{x+1} & \text{if} \quad q < p, \end{cases} \qquad \text{with} \quad \gamma = q/p,$$

is positive on $\mathbb{N}$ and satisfies $P^+h = h$. This readily implies that the operator P_h^+ defined by

$$\forall \varphi : \mathbb{N} \to \mathbb{R}^+, \quad P_h^+(\varphi) = \frac{1}{h}P^+(h\varphi)$$

is markovian on $\mathbb{N}$. It is called the "**Doob transform**" of the operator P^+. More precisely, $P_h^+ = (p_h(x,y))_{x,y\geq 0}$ with:

• if $p = 1/2$,

$$p_h(x,y) = \begin{cases} 1 & \text{for } x = 0, y = 1, \\ \frac{x}{2(x+1)} & \text{for } x \geq 1, y = x-1, \\ \frac{x+2}{2(x+1)} & \text{for } x \geq 1, y = x+1, \\ 0 & \text{otherwise,} \end{cases}$$

• if $p \neq 1/2$, setting $\gamma = q/p$,

$$p_h(x,y) = \begin{cases} 1 & \text{for } x = 0, y = 1, \\ q\ \frac{\gamma^x - 1}{\gamma^{x+1}-1} & \text{for } x \geq 1, y = x-1, \\ p\ \frac{\gamma^{x+2} - 1}{\gamma^{x+1}-1} & \text{for } x \geq 1, y = x+1, \\ 0 & \text{otherwise,} \end{cases}$$

Notice that, as conjugation in linear algebra or group theory, it holds

$$p_h(x_0, \ldots, x_n) := p_h(x_0, x_1) \ldots p_h(x_{n-1}, x_n) = \frac{h(x_n)}{h(x_0)} p(x_0, \ldots, x_n).$$

3.2 On the process $(S_n - 2\mathbf{m}_n)_{n\geq 0}$ and the Pitman's theorem

We consider in this section the process $(W_n)_{n\geq 0} = (S_n - 2\mathbf{m}_n)_{n\geq 0}$.

The equalities $S_{d_k} \mathbf{1}_{d_k < +\infty} = -k$ $\mathbb{P}$-a.s. for any $k \geq 0$ yield

$$(S_{d_k} - 2\mathbf{m}_{d_k})\ \mathbf{1}_{d_k < +\infty} = k \qquad \mathbb{P}\text{-a.s.}$$

The decompositions (2.1) and (2.2) of the trajectories of $(S_n)_{n\geq 0}$ transfer to those of $(W_n)_{n\geq 0}$:
• when $p \leq 1/2$, the sequence $W_0, W_1, W_2, \ldots$ may be decomposed as

$$\underbrace{S_0, S_1, \ldots, S_{d_1 - 1} = 0}_{=:\mathcal{W}_1}, \underbrace{2 + S_{d_1}, 2 + S_{d_1+1}, \ldots, 2 + S_{d_2 - 1}}_{=:\mathcal{W}_2}, \quad \ldots\ldots$$

$$\ldots\ldots \quad \underbrace{2k + S_{d_k}, 2k + S_{d_k+1}, \ldots, 2k + S_{d_{k+1}-1}}_{=:\mathcal{W}_k}, \quad \ldots$$

The r.v. $\mathcal{W}_1, \mathcal{W}_2, \ldots$ are iid. Furthermore, for any finite sequence $\mathbf{x} = (x_0, x_1, x_2, \ldots, x_n)$ in $\mathbb{N}$ with $x_0 = x_{n-1} = 0, x_n = 1$ and $|x_i - x_{i-1}| = 1$ for $1 \leq i \leq n-1$, it holds

$$\mathbb{P}(\mathcal{W}_1 = \mathbf{x}) = \mathbb{P}(\mathcal{D}_1 = (x_0, \ldots, x_{n-1} =, -1)) = p(x_0, \ldots, x_n) \times \gamma$$

where $\gamma = q/p$.

• When $p > 1/2$, the sequence $W_0, W_1, W_2, \ldots$ may be decomposed as

$$\underbrace{S_0, S_1, \ldots, S_{d_1-1}}_{=:\mathcal{W}_1}, \quad \ldots\ldots \quad , \underbrace{2(\kappa - 1) + S_{d_{\kappa-1}}, \ldots, 2(\kappa-1) + S_{d_\kappa - 1}}_{=:\mathcal{W}_{\kappa-1}},$$

$$\underbrace{2\kappa + S_{d_\kappa}, 2\kappa + S_{d_\kappa + 1}, 2\kappa + S_{d_\kappa+2}, \quad \ldots}_{=:\mathcal{W}_\infty}$$

The random variables $\mathcal{W}_1 \mathbf{1}_{d_1 < +\infty}, \mathcal{W}_2 \mathbf{1}_{d_2 < +\infty}, \ldots$ are also iid and the last variable $\mathcal{W}_\infty$ has same distribution as $\mathcal{W}_1 \mathbf{1}_{d_1 = +\infty}$. As previously, for any finite sequence $\mathbf{x} = (x_0, x_1, x_2, \ldots, x_n)$ in $\mathbb{N}$ with $x_0 = x_{n-1} = 0, x_n = 1$ and $|x_i - x_{i-1}| = 1$ for $1 \leq i \leq n-1$, it holds

$$\mathbb{P}(\mathcal{W}_1 \mathbf{1}_{d_1 < +\infty} = \mathbf{x}) = p(x_0, \ldots, x_n) \times \gamma$$

while, for any infinite sequence $\mathbf{x}$ in $\mathbb{N}$ with $x_0 = 0$ and $|x_i - x_{i-1}| = 1$ for $i \geq 1$,

$$\mathbb{P}(\mathcal{W}_1 \mathbf{1}_{d_1=+\infty} = \mathbf{x}) = \mathbb{P}(\mathcal{D}_1 \mathbf{1}_{d_1=+\infty} = \mathbf{x}).$$

The goal of these pages is to prove the following statement.

Theorem 3.1. *The process $(S_n - 2\mathbf{m}_n)_{n\geq 0}$ is a Markov chain on $\mathbb{N}$ with one step transition probability operator P_h^+.*

To prove this statement, it suffices to check that, for any finite sequence $\mathbf{x} = (x_0, \ldots, x_n)$ in $\mathbb{N}$ such that $x_0 = 0$ and $|x_\ell - x_{\ell+1}| = 1$ for $0 \leq \ell \leq n-1$, it holds

$$\mathbb{P}(W_\ell = x_\ell, 0 \leq \ell \leq n) = \prod_{\ell=0}^{n-1} p_h(x_\ell, x_{\ell+1}) = \frac{h(x_n)}{h(x_0)} p(x_0, \ldots, x_n).$$

The set $\Lambda := (W_\ell = x_\ell, 0 \leq \ell \leq n)$ may be decomposed as the disjoint union of the events $\Lambda_x, 0 \leq x \leq x_n$, defined by

$$\Lambda_x := \Lambda \cap \left(\min_{i\geq n}(W_i) = x \right)$$

so that, we may write $\mathbb{P}(\Lambda) = \sum_{x=0}^{x_n} \mathbb{P}(\Lambda_x) = \sum_{x=0}^{x_n} \sum_{k=0}^{+\infty} \mathbb{P}(\Lambda_x, d_k \leq n < d_{k+1})$.

Furthermore, for any $k \geq 0$, it holds $\min_{i\geq d_k}(W_i) \geq k$; consequently

$$\left(\min_{i\geq n}(W_i) = x \right) \cap (d_k \leq n < d_{k+1}) = \emptyset \quad \text{if} \quad k > x,$$

so that $\mathbb{P}(\Lambda)$ may be decomposed as

$$\mathbb{P}(\Lambda) = \sum_{x=0}^{x_n} \sum_{k=0}^{x} \mathbb{P}(\Lambda_x, d_k \leq n < d_{k+1}). \tag{3.1}$$

3.3 Proof of Pitman's theorem when $p = 1/2$

In this cases, all the $d_k, k \geq 0$, are $\mathbb{P}$-as finite and $\min_{i\geq d_{k+1}}(W_i) = k+1$. Furthermore

$$(d_k \leq n < d_{k+1}) \Longrightarrow \min_{i\geq n}(W_i) = k.$$

Hence

$$\Lambda_x \cap (d_k \leq n < d_{k+1}) = \begin{cases} \emptyset & \text{if} \quad k \neq x, \\ \Lambda_x & \text{if} \quad k = x \end{cases}.$$

Consequently, equation (3.1) becomes

$$\begin{aligned}
\mathbb{P}(\Lambda) &= \sum_{x=0}^{x_n} \mathbb{P}(\Lambda_x) \\
&= \sum_{x=0}^{x_n} \mathbb{P}(W_0 = x_0, \ldots, W_n = x_n, \min_{i\geq n}(W_i) = x) \\
&= \sum_{x=0}^{x_n} \gamma^x \times p(x_0, \ldots, x_n) \ \underbrace{\mathbb{P}\left(\min_{i\geq n}(W_i) = x \right)}_{=\mathbb{P}(\exists i\geq 1 | x_n + S_i = x-1) = 1} \quad \text{with} \quad \gamma = 1, \\
&= p(x_0, \ldots, x_n)(x_n + 1) = \frac{h(x_n)}{h(0)} p(x_0, \ldots, x_n) = p_h(x_0, \ldots, x_n).
\end{aligned}$$

3.4 Proof of Pitman's theorem when $p < 1/2$

In this cases, all the $d_k, k \geq 0$, are still $\mathbb{P}$-as finite and $\min_{i \geq d_{k+1}} (W_i) = k+1$. As above,

$$\Lambda_x \cap (d_k \leq n < d_{k+1}) = \begin{cases} \emptyset & \text{if} \quad k \neq x, \\ \Lambda_x & \text{if} \quad k = x \end{cases}$$

and equation (3.1) becomes

$$\begin{aligned} \mathbb{P}(\Lambda) &= \sum_{x=0}^{x_n} \mathbb{P}(\Lambda_x) \\ &= \sum_{x=0}^{x_n} \gamma^x p(x_0, \ldots, x_n) \ \underbrace{\mathbb{P}\left(\min_{i \geq n}(W_i) = x\right)}_{=\mathbb{P}(\exists i \geq 1 | x_n + S_i = x-1) = 1} \\ &= p(x_0, \ldots, x_n) \frac{\gamma^{x_n+1} - 1}{\gamma - 1} \\ &= \frac{h(x_n)}{h(x_0)} p(x_0, \ldots, x_n) = p_h(x_0, \ldots, x_n). \end{aligned}$$

3.5 Proof of Pitman's theorem when $p > 1/2$

This is the most interesting case since $\mathbb{P}(d_1 = +\infty) > 0$.
We start again with equation (3.1)

$$\mathbb{P}(\Lambda) = \sum_{x=0}^{x_n} \sum_{k=0}^{x} \mathbb{P}(\Lambda_x, d_k \leq n < d_{k+1}).$$

notice that $W_{d_{k+1}-1} \mathbf{1}_{d_k < +\infty} = k$; hence,

$$0 \leq k < x \quad \Longrightarrow \quad \Lambda_x \cap (d_k \leq n < d_{k+1}) \ \subset (d_{k+1} = +\infty)$$

Consequently,

- <u>when $0 \leq k < x$</u>,

$$\begin{aligned} \mathbb{P}(\Lambda_x, d_k \leq n < d_{k+1}) &= \mathbb{P}(\Lambda_x, d_k \leq n < d_{k+1} = +\infty) \\ &= p(x_0, \ldots, x_n) \ \gamma^k \ \mathbb{P}\left(\min_{i \geq n}(W_i) = x\right). \end{aligned}$$

- <u>when $k = x$</u>,

$$\begin{aligned} \mathbb{P}(\Lambda_x, d_k \leq n < d_{k+1}) &= \mathbb{P}(\Lambda_x, d_x \leq n < d_{x+1} = +\infty) \\ &\quad + \mathbb{P}(\Lambda_x, d_x \leq n < d_{x+1} < +\infty) \\ &= p(x_0, \ldots, x_n) \ \gamma^x \ \mathbb{P}\left(\min_{i \geq n}(W_i) = x\right) \\ &\quad + p(x_0, \ldots, x_n) \ \gamma^x \ \mathbb{P}(\exists i \geq 1 \mid x_n + S_i = x - 1). \end{aligned}$$

Consequently,

$$\mathbb{P}(\Lambda) = p(x_0,\ldots,x_n)\sum_{x=0}^{x_n}\left(\sum_{k=0}^{x} \gamma^k\ \mathbb{P}\left(\min_{i\geq n}(W_i)=x\right)\right)$$
$$+\, p(x_0,\ldots,x_n)\ \sum_{x=0}^{x_n}\gamma^x\ \mathbb{P}(\exists i\geq 1 \mid x_n+S_i = x-1). \qquad (3.2)$$

By lemma 4.2, it holds

$$\mathbb{P}(\exists i\geq 1 \mid x_n+S_i=x-1) = \mathbb{P}_{x_n-x+1}(\tau_0<+\infty) = \gamma^{x_n-x+1}$$

and

$$\begin{aligned}
\mathbb{P}\left(\min_{i\geq n}(W_i)=x\right) &= \mathbb{P}\Big((\exists i\geq 1 \mid x_n+S_i=x)\cap(\forall i\geq 1 \mid x_n+S_i\neq x-1)\Big)\\
&= \mathbb{P}\Big((\exists i\geq 1 \mid x_n+S_i=x)\setminus(\exists i\geq 1 \mid x_n+S_i=x-1)\Big)\\
&= \mathbb{P}_{x_n-x}(\tau_0<+\infty)-\mathbb{P}_{x_n-x+1}(\tau_0<+\infty)\\
&= (1-\gamma)\gamma^{x_n-x}.
\end{aligned}$$

Finally, equation (3.2) yields

$$\begin{aligned}
\mathbb{P}(\Lambda) &= p(x_0,\ldots,x_n)\sum_{k=0}^{x_n}\gamma^k\\
&= p(x_0,\ldots,x_n)\frac{1-\gamma^{x_n+1}}{1-\gamma}\\
&= p(x_0,\ldots,x_n)\frac{h(x_n)}{h(0)} = p_h(x_0,\ldots,x_n).
\end{aligned}$$

4 Auxiliary results

4.1 Harmonic functions for the killed random walk on $\mathbb{N}$

We consider the submarkovian transition operator P^+ on $\mathbb{N}$ given by: for any $x\in\mathbb{N}$ and any function $\varphi:\mathbb{N}\to\mathbb{R}^+$,

$$P^+\varphi(x) := \mathbb{E}_x(\varphi(S_1), S_1\geq 0) = \mathbb{E}_x(\varphi(S_1), d_1\geq 1).$$

In other words, $P^+=(p^+(x,y))_{x,y\in\mathbb{N}}$ with $p^+(x,y)=p(x,y)\mathbf{1}_{\mathbb{N}}(x)\mathbf{1}_{\mathbb{N}}(y)$:

$$P^+ = \begin{pmatrix} 0 & p & 0 & 0 & \ldots\\ q & 0 & p & 0 & \ldots\\ 0 & q & 0 & p & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Let us describe the set of P^+-harmonic functions $\mathcal{H}(P^+)$ on $\mathbb{N}$.

Lemma 4.1. *Let $h=(h(x))_{x\geq 0}$ be a P^+-harmonic function on $\mathbb{N}$. Then, there exists $a\in\mathbb{R}$ such that*

- *if $p=q=1/2$ then $h(x)=a(x+1)$ for any $x\geq 0$;*
- *if $p\neq q$ then $h(x)=a(1-(q/p)^{x+1})$ for any $x\geq 0$.*

Proof. The equality $P^+h = h$ yields $\forall x \geq 1, \quad h(x) = qh(1) + ph(x+1) \quad \text{and} \quad h(0) = ph(1)$. The corresponding characteristic equation is $pr^2 - r + q = 0 = (r-1)(pr-q)$. Hence,

- if $p = q = 1/2$ then $h(x) = ax + b$ with $a, b \in \mathbb{R}$. The equality $h(0) = h(1)/2$ yields $b = a$.
- if $p \neq q$ then $h(x) = a + b(q/p)^x$ with $a, b \in \mathbb{R}$. The equality $h(0) = ph(1)$ yields $b = -aq/p$.

□

4.2 On the hitting probability of the origin

Let $\tau_0 : \Omega \to \mathbb{N}$ be the hitting time of the origin defined by $\tau_0 := \inf\{n \geq 0 \mid S_n = 0\}$. The hitting probability of the origin, starting from $x \geq 0$ equals $\mathbb{P}_x(\tau_0 < +\infty)$.

Lemma 4.2. *For any $x \geq 0$, it holds*
- *if $p \leq 1/2$ then $\mathbb{P}_x(\tau_0 < +\infty) = 1$;*
- *if $p > 1/2$ then $\mathbb{P}_x(\tau_0 < +\infty) = (q/p)^x$.*

The proof is similar to the one of Lemma 4.1 but the boundary condition is different. We consider the Markov chain $(X'_n)_{n\geq 0}$ on $\mathbb{N}$ with probability transition $(p'(x,y))_{x,y}$ given by $p'(x,y) = p(x,y)$ when $x \geq 1$ and $p'(0,0) = 1$. The function $g : x \mapsto \mathbb{P}_x(\tau_0 < +\infty)$ is P'-harmonic on $\mathbb{N}$:

$$\forall x \geq 1, \quad g(x) = qg(x-1) + pg(x+1) \quad \text{and} \quad g_0 = 1.$$

Consequently, there exist $a, b \in \mathbb{R}$ such that $g(x) = ax + b$ when $p = q = 1/2$ while $g(x) = a(q/p)^x + b$ when $p \neq q$. The condition $g_0 = 1$ yields

$$g(x) = \begin{cases} ax + 1 & \text{if} \quad p = 1/2 \\ a((q/p)^x - 1) + 1 & \text{if} \quad p \neq q. \end{cases}$$

- $p \leq 1/2$.
  The fact that g is bounded yield $g(x) = 1$ (for any $x \geq 0$).
- $p > 1/2$.
  It holds $S_n \to +\infty$ $\mathbb{P}$-as, hence $\min(S_n, n \geq 0) > -\infty$ $\mathbb{P}$-as which yields

$$\mathbb{P}(\min(S_n) \geq -x) \nearrow 1 \quad \text{as} \quad x \to +\infty.$$

Consequently $g(x) = \mathbb{P}_x(\tau_0 < +\infty) = \mathbb{P}(x + \min(S_n) \leq 0) \searrow 0$ as $x \to +\infty$.
Since $q/p < 1$, we obtain $a = 1$ hence $g(x) = (q/p)^x$ for any $x \geq 0$.

□

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