Some probabilistic aspects of representation theory

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1 Abstract

The mini-course will be concerned with interactions between representation theory of simple Lie algebras over $\mathbb{C}$ and certain random walks defined on lattices in Euclidean spaces. We will essentially restrict ourself to the so called “ballot random walk” which is related to the representation theory of $\mathfrak{sl}_n$. Let $B = (e_1, \ldots, e_n)$ be the standard basis of $\mathbb{R}^n$. The ballot random walk W can be defined as the Markov chain with seps in the base B. We will review some results due to O'Connell [9],[10] giving the law of the random walk conditioned to never exit the cone

$$\overline{C} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq \cdots \geq x_n \geq 0\}$$

and its probability to never exit $\overline{C}$. This is achieved in [9] by considering first a natural transformation $\mathcal{P}$ which associates to any path with steps in B a path in the Weyl chamber $\overline{C}$, next by checking that the image of the random walk W by this transformation is a Markov chain and finally by establishing that this Markov chain has the same law as W conditioned to never exit $\overline{C}$. The transformation $\mathcal{P}$ is based on the Robinson-Schensted-Knuth correspondence which maps the words on the ordered alphabet $\mathcal{A}_n = \{1 < \cdots < n\}$ (regarded as finite paths in $\mathbb{R}^n$) on pairs of semistandard tableaux and can be regarded as a n-dimensional analogue of the celebrated Pitman transform. Here we will expose simpler proofs of O'Connell's results based on the reflection principle of Gessel and Zeilberger [4].

The results of O'Connell can be extended to a large class of random walks defined from representations of Lie algebras (or their generalizations known as Kac-Moody algebras). The corresponding transformation $\mathcal{P}$ is then defined in terms of the Littelmann path model. In the opposite direction, the probabilistic approach can be used to obtain asymptotic behavior of tensor product multiplicities. We refer the interested reader to [1], [5], [6] and [7] for a complete exposition that we shall not detail in this mini-course since it requires more specialized material.

If time permits, we will present another application of the RSK-correspondence in the study of certain percolation models in random matrices whose entries follow independent geometric laws. This type of model has been deeply studied by Johansson in [3], who proved that the fluctuations of the previous last passage percolation, once correctly normalized, are controlled by the Tracy-Widom distribution.

The minicourse will be as self-containing as possible, in particular all the probabilistic, combinatorial and algebraic notions needed will be recalled. We will begin with the study of the simple random walk in dimension 1 and obtain the probability that it remains nonnegative.

This will be then generalized to higher dimension. We will next define the transformation $\mathcal{P}$ and prove that $\mathcal{P}(W)$ has the same law as W conditioned to stay in $\overline{C}$. We will then turn to the exposition of the previous last passage percolation model.

References

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