Lévy and Pitman representation theorems

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Abstract

The Lévy representation theorem establishes that a reflected Brownian motion, which is the absolute value of a Brownian motion denoted by $|B(t)|$, is equal in law to a Brownian motion $W(t)$ minus its infimum before time t, denoted by $I(t) := \inf_{s\leqslant t} W(s)$. More precisely, if $L(t) := \lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_0^t \mathbb{1}_{(0,\epsilon)}(B(s))\mathrm{d}s$ represents the local time at 0 of the Brownian motion B, this theorem provides the equality in law of the following two pairs of processes:

$$(|B(t)|, L(t); t \geqslant 0) \overset{\text{(law)}}{=} (W(t) - I(t), -I(t); t \geqslant 0).$$

Pitman's representation theorem establishes that a Brownian motion conditioned to stay positive, which is a three-dimensional Bessel process denoted by R_t, is equal in law to a Brownian motion $W(t)$ minus twice its infimum $I(t)$. More precisely, if $J_t := \inf_{s\geqslant t} R_s$ is the infimum of the Bessel process R after time t, this theorem provides the equality in law of the following two pairs of processes:

$$(R(t), J(t); t \geqslant 0) \overset{\text{(law)}}{=} (W(t) - 2I(t), -I(t); t \geqslant 0).$$

In this presentation, we will start by introducing some fundamental notions of stochastic calculus such as local times, Bessel processes, and Doob's h-transformations. These will allow us to present these two theorems and sketch their proofs.

This presentation offers a continuous one-dimensional version of the results presented by Marc Peigné in his talk on a discrete version of Pitman's theorem for random walks in $\mathbb{Z}^d$.

Contents

Preface

This course is inspired by the references cited in the bibliography, the main one being the book by Yen and Yor [6]. The aim of this mini-course is to provide a heuristic and pedagogical approach to Lévy and Pitman theorems, intentionally leaving aside certain technical aspects. For a more rigorous and comprehensive approach, an interested reader may refer to the bibliography.

Feel free to report any remaining errors or typos by email, as well as to propose any suggestions aimed at improving this mini-course: `sandro.franceschi@telecom-sudparis.eu`.

1 Brownian Motion and Local Time

Brownian Motion and Donsker's Theorem

Definition 1.1 (Standard Brownian Motion)**.** A *standard Brownian motion* is a real process (W_t) which is:

i) *continuous in its paths* (i.e., $t \mapsto W_t$ is continuous);

ii) has *independent increments* (i.e., for any $n \geqslant 2$ and any $0 \leqslant t_1 < \cdots < t_n$, the family $(W_{t_1}, W_{t_2} - W_{t_1}, \cdots, W_{t_n} - W_{t_{n-1}})$ is independent, i.e., $\forall s < t$, $W_t - W_s$ is independent of $\mathcal{F}_s$);

iii) has *stationary increments* (i.e., $W_t - W_s$ has the same law as W_{t-s} for all $s \leqslant t$);

iv) and for all $t \geqslant 0$, W_t follows a Gaussian distribution $\mathcal{N}(0, t)$.

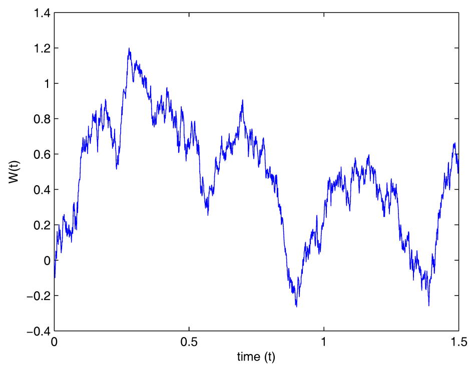

Figure 1: Simulation of a standard Brownian motion trajectory

Approximation via Random Walks It is possible to construct Brownian motion as a limit of rescaled random walks.

Consider $X_1, \cdots, X_n$ as a sequence of independent and identically distributed random variables, following a Bernoulli distribution with parameter $1/2$, such that $\mathbb{P}(X_k = 1) = \mathbb{P}(X_k = -1) = 1/2$. The symmetric simple random walk is defined as:

$$S_n = X_1 + \cdots + X_n$$

which satisfies $\mathbb{E}(S_n) = 0$ and $\mathbb{V}\mathrm{ar}(S_n) = n$. To transform this into a continuous function, define $S_0 = 0$ and for all $t > 0$,

$$S_t := S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor) X_{\lfloor t \rfloor + 1}.$$

Then, perform a double renormalization:

- in *time* (with a factor $1/n$),
- in *space* (with a factor $1/\sqrt{n}$),

and define for all $t \in [0,1]$ the process:

$$S_t^{(n)} := \frac{S_{nt}}{\sqrt{n}} = \frac{1}{\sqrt{n}} \left(S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1} \right).$$

Theorem 1.2 (Donsker's Theorem)**.** *The sequence of processes $(S_t^{(n)})$ converges in law to a standard Brownian motion (W_t) in $\mathscr{C}([0,1],\mathbb{R})$.*

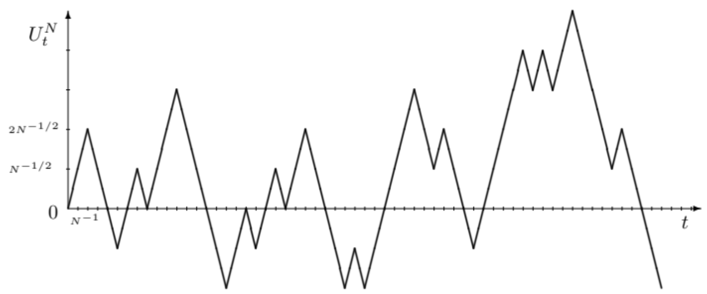

Figure 2: Illustration of Donsker's Theorem

Remark 1.3 (Discrete Version of Lévy and Pitman Theorems)**.** To prove the Lévy and Pitman representation theorems, it is possible to use the discrete version of these theorems via Donsker's theorem, which allows approximating Brownian motion with a random walk, and a similar (but more complex!) theorem that approximates the three-dimensional Bessel process. In this document, we will detail a possible strategy to prove these theorems without resorting to their discrete version or the application of limit theorems.

Itô's Formula

If X_t were a differentiable process (which is generally not the case!), we would have the formula $(f \circ X)' = (f' \circ X)X'$, and by the classical integration formula:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)X_s' \mathrm{d}s = f(X_0) + \int_0^t f'(X_s)\mathrm{d}X_s.$$

However, this formula is FALSE in the general case of a diffusion. Due to the quadratic nature of the martingale part of X, the second-order term in Taylor's expansion is needed to obtain a correct formula. This is what Itô's formula accomplishes!

Theorem 1.4 (Itô's Formula)**.** *Let (X) be an Itô process and f a $\mathcal{C}^2$ function. Then:*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s)\mathrm{d}\langle X\rangle_s.$$

where $\langle X \rangle$ is the quadratic variation of X.

For a standard Brownian motion B, this simply becomes:

$$f(B_t) = f(0) + \int_0^t f'(B_s)\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s)\mathrm{d}s.$$

Local Time and Tanaka's Formula

Let (B_t) be a standard real Brownian motion. We aim to study the time the process spends at a point $a \in \mathbb{R}$. It is pointless to consider $\int_0^t \mathbb{1}_{B_s=a}\mathrm{d}s$, as this integral is zero:

$$\mathbb{E}\left(\int_0^t \mathbb{1}_{B_s=a}\mathrm{d}s\right) = \int_0^t \mathbb{P}(B_s = a)\mathrm{d}s = 0.$$

The meaningful concept is the occupation time density. For more details on the results in this section, refer to [5] and [6].

Definition 1.5 (Occupation Measure and Local Time)**.** The occupation measure μ_t is defined as:

$$\mu_t(A) = \int_0^t \mathbb{1}_A(B_s)\mathrm{d}s,$$

for any measurable set $A \subset \mathbb{R}$. It represents the time spent by the process in A before time t. This measure admits a density $L_t^\bullet$ with respect to the Lebesgue measure, such that $\mu_t(\mathrm{d}a) = L_t^a \mathrm{d}a$, i.e.,

$$\mu_t(A) = \int_A L_t^a \mathrm{d}a.$$

This density, called the local time, is defined for all $a \in \mathbb{R}$ by:

$$L_t^a = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \mathbb{1}_{\{a \leqslant B_s < a+\epsilon\}} \mathrm{d}s.$$

Proposition 1.6 (Occupation Time Formula)**.** *Almost surely, the function $(t, a) \mapsto L_t^a$ is continuous and increasing in t. Moreover, $t \mapsto L_t^a$ increases only when $B_t = a$, i.e., the support of $\mathrm{d}_t L_t^a$ coincides with $\{t \geqslant 0 : B_t = a\}$. For any measurable positive function f, almost surely:*

$$\int_0^t f(B_s)\mathrm{d}s = \int_{\mathbb{R}} f(a) L_t^a \mathrm{d}a.$$

This equality transforms a temporal integral into a spatial integral.

The concept of reflection is closely tied to local times. In one dimension, reflection is achieved simply via the absolute value.

Proposition 1.7 (Tanaka's Formula)**.** *Let B_t be the standard Brownian motion and L_t^a its local time at $a \in \mathbb{R}$. Then:*

$$|B_t - a| = |B_0 - a| + \int_0^t \operatorname{sgn}(B_s - a)\mathrm{d}B_s + L_t^a.$$

Proof. Apply (formally and incorrectly) Itô's formula to the absolute value function (which is not $\mathcal{C}^2$; approximation is necessary for rigor). Let $f = |\cdot - a|$. Then $f' = \operatorname{sgn}(\cdot - a)$ and $f'' = 2\delta_a$, where δ_a is the Dirac delta at a. If Itô's theorem were applicable, we would have:

$$|B_t - a| = |B_0 - a| + \int_0^t \operatorname{sgn}(B_s - a)\mathrm{d}B_s + \frac{1}{2}\int_0^t 2\delta_a(B_s)\mathrm{d}s.$$

Noting that, in the distributional sense:

$$L_t^a = \int \delta_a(u) L_t^u \mathrm{d}u = \int_0^t \delta_a(B_s)\mathrm{d}s,$$

the desired formula follows. □

Remark 1.8 (Local Time and Reflection at 0)**.** The process $W_t := \int_0^t \operatorname{sgn}(B_s - a)\mathrm{d}B_s$ is a Brownian motion according to Lévy's characterization, since

$$\langle \int_0^\cdot \operatorname{sgn}(B_s - a)\mathrm{d}B_s \rangle_t = \int_0^t (\operatorname{sgn}(B_s - a))^2 \mathrm{d}\langle B \rangle_s = \int_0^t \mathrm{d}s = t.$$

Setting $a = 0$, we obtain the decomposition:

$$|B_t| = W_t + L_t^0. \tag{1}$$

Thus, the local time at 0 can be interpreted as an infinitely strong drift that increases only when the process hits 0, preventing it from crossing 0 and forcing it to reflect off this boundary.

We now state the following intuitive result, which will be used in the proof of Pitman's theorem.

Proposition 1.9 (Time Reversal and Local Time)**.** *Let $\tau_1 = \inf\{t : L_t \geqslant 1\}$. Then:*

$$(B_{\tau_1 - t}; t \leqslant \tau_1) \overset{\text{(law)}}{=} (B_t; t \leqslant \tau_1).$$

Proof. Since the Brownian bridge is stable under time inversion, the proof follows from the result on pseudo-Brownian bridges (see Example 4.3.1(b), Section 4.3 in Yen and Yor [6]). Alternatively, one can directly observe that:

$$\mathbb{E}(F(B_u; u \leqslant \tau_1) \,|\, \tau_1 = t) = \mathbb{E}(F(B_u; u \leqslant t) \,|\, B_t = 0, L_t = 1).$$

□

2 Lévy's Representation Theorem for Reflected Brownian Motion

We now state Skorokhod's Lemma, which is used in the proof of Lévy's representation theorem. However, this result also has intrinsic value as it provides insight into the local time and the phenomenon of reflection.

Lemma 2.1 (Skorokhod's Lemma)**.** *Let w be a continuous function defined on $\mathbb{R}_+$ such that $w(0) = 0$. There exists a unique pair of functions (x, l) satisfying:*

- *$x = w + l$,*
- *x is nonnegative,*
- *l is increasing, vanishes at 0, and $\mathrm{d}l_s$ has support $\{s : x(s) = 0\}$.*

Moreover, l is given by:

$$l(t) = -\inf_{s \leqslant t}(w(s)).$$

Proof. It is straightforward to verify that the pair $(x, l) = (w - \inf_{s\leqslant t}(w(s)), -\inf_{s\leqslant t}(w(s)))$ satisfies the three conditions above. Now that existence is established, let us prove uniqueness by considering two solutions (x_1, l_1) and (x_2, l_2). Using the Stieltjes integral, we obtain:

$$\begin{aligned} 0 \leqslant (x_1(u) - x_2(u))^2 = (l_1(u) - l_2(u))^2 &= 2\int_0^u (x_1(v) - x_2(v))\,(\mathrm{d}l_1(v) - \mathrm{d}l_2(v)) \\ &= -2\int_0^u x_1(v)\mathrm{d}l_2(v) - 2\int_0^u x_2(v)\mathrm{d}l_1(v) \leqslant 0, \end{aligned}$$

which implies $x_1 = x_2$ and $l_1 = l_2$. □

The following identity by Lévy characterizes the law of the local time at 0 for a Brownian motion.

Theorem 2.2 (Lévy's Representation Theorem)**.** *Consider B_t a standard Brownian motion and L_t^0 its local time at 0, as well as W_t a standard Brownian motion and $I_t = \inf_{s\leqslant t}(W_s)$ its infimum. Then, the following equality in law holds:*

$$(W_t - I_t, -I_t; t \geqslant 0) \overset{\textit{(law)}}{=} (|B_t|, L_t^0; t \geqslant 0).$$

Proof. Tanaka's formula and the properties of local time imply that there exists a standard Brownian motion W_t such that:

$$|B_t| = W_t + L_t^0,$$

where $|B_t|$ is nonnegative, and L_t^0 is increasing and grows only when $|B_t| = 0$. Skorokhod's lemma then implies:

$$L_t^0 = -\inf_{s\leqslant t}(W_s).$$

Thus, the following almost sure equality holds:

$$(W_t - I_t, -I_t; t \geqslant 0) = (|B_t|, L_t^0; t \geqslant 0),$$

leading to the equality in law stated in the theorem. □

In fact, we have shown that for a well-chosen W, the equality holds almost surely and not just in law!

Remark 2.3 (Alternative Formulation of Lévy's Representation Theorem)**.** Let $S_t = \sup_{s\leqslant t}(W_s)$. Then:

$$(S_t - W_t, S_t; t \geqslant 0) \overset{\text{(law)}}{=} (|B_t|, L_t^0; t \geqslant 0).$$

Proposition 2.4 (Joint Law)**.** *The joint law of $(|W_t|, L_t^0)$ is given by:*

$$\mathbb{P}(|B_t| \in \mathrm{d}x, L_t^0 \in \mathrm{d}y) = 2\frac{x+y}{\sqrt{2\pi t^3}}e^{-\frac{(x+y)^2}{2t}}\mathbb{1}_{x\geqslant 0, y\geqslant 0}\mathrm{d}x\mathrm{d}y.$$

Proof. Using the reflection principle (Proposition 2.5), one can show that $\mathbb{P}(S_t \geqslant a, B_t \geqslant b) = \mathbb{P}(B_t \geqslant 2a + b)$ for all $a \geqslant 0$ and $-b \leqslant a$. With some variable transformations, the joint law of $(|W_t|, L_t^0)$ follows via Lévy's identity. □

Proposition 2.5 (Reflection Principle/Désiré André's Theorem)**.** *Let $\{W_t\}_{t\geqslant 0}$ be a Brownian motion, and let $t > 0$. Define:*

$$S_t := \sup_{0\leqslant s\leqslant t} W_s.$$

For $x \geqslant 0$ and $y \leqslant x$, we have:

$$\mathbb{P}(S_t \geqslant x, W_t \leqslant y) = \mathbb{P}(W_t \geqslant 2x - y).$$

Proof. A diagram is necessary to understand the heuristic. The idea is to use the strong Markov property. First, we write:

$$\mathbb{P}\left[S_t \geqslant x, W_t \leqslant y\right] = \mathbb{P}\left[T_x \leqslant t, W_t \leqslant y\right] = \mathbb{P}\left[T_x \leqslant t, W_{t-T_x+T_x} - W_{T_x} \leqslant y - W_{T_x}\right].$$

On the one hand, $W_{T_x} = x$, due to the continuity of W. On the other hand, $W_s^x = W_{s+T_x} - W_{T_x}$ is a Brownian motion independent of $\mathscr{F}_{T_x}$ (and thus of T_x). We deduce that:

$$\mathbb{P}\left[S_t \geqslant x, W_t \leqslant y\right] = \mathbb{P}\left[T_x \leqslant t, W^x_{t-T_x} \leqslant y - x\right] = \mathbb{P}\left[T_x \leqslant t, W^x_{t-T_x} \geqslant x - y\right],$$

where the second equality comes from the symmetry of the process W^x given T_x. But:

$$\mathbb{P}\left[T_x \leqslant t, W^x_{t-T_x} \geqslant x - y\right] = \mathbb{P}\left[T_x \leqslant t, W_t - x \geqslant x - y\right] = \mathbb{P}\left[W_t \geqslant 2x - y\right],$$

where the second equality follows from the inclusion $\{W_t \geqslant 2x - y\} \subset \{T_x \leqslant t\}$, which immediately arises from $2x - y \geqslant x$. □

3 Girsanov's Transformation and Doob's h-Transform

Change of Measure and Girsanov's Theorem

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A probability $\mathbb{Q}$ on $(\Omega, \mathcal{A})$ is absolutely continuous with respect to $\mathbb{P}$ if and only if:

$$\forall A \in \mathcal{A}, \quad (\mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A) = 0).$$

We note $\mathbb{Q} \ll \mathbb{P}$ and we also say that $\mathbb{Q}$ is dominated by $\mathbb{P}$. Furthermore, $\mathbb{P}$ and $\mathbb{Q}$ are said to be *equivalent* if each is absolutely continuous with respect to the other.

Theorem 3.1 (Radon Nikodym)**.** *$\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ if and only if there exists a nonnegative random variable Z on $(\Omega, \mathcal{A})$ such that:*

$$\forall A \in \mathcal{A}, \quad \mathbb{Q}(A) = \int_A Z(\omega)\mathrm{d}\mathbb{P}(\omega).$$

Z is called the density of $\mathbb{Q}$ with respect to $\mathbb{P}$ (or the Radon-Nikodym derivative *of $\mathbb{Q}$ with respect to $\mathbb{P}$) and is sometimes denoted by $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$.*

If $\mathbb{Q}$ is dominated by $\mathbb{P}$, then $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if and only if $\mathbb{P}(Z > 0) = 1$.

Let $\mathbb{P}, \mathbb{Q}$ be two probabilities on a measurable space equipped with a filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0})$. If $\mathbb{Q} \ll \mathbb{P}$ on $\mathcal{F}_\infty = \vee_{t\geqslant 0}\mathcal{F}_t$, we define:

$$Z_\infty := \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_\infty}.$$

For any $t > 0$, $\mathbb{Q} \ll \mathbb{P}$ on $\mathcal{F}_t$, and we denote:

$$Z_t := \mathbb{E}_{\mathbb{P}}\left[Z_\infty \,|\, \mathcal{F}_t\right] = \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}.$$

(We say that the process $(Z_t)_{t\geqslant 0}$ is a uniformly integrable martingale under $\mathbb{P}$, closed by Z_∞). In what follows, we assume that: $(Z_t)_{t\geqslant 0}$ is a continuous $\mathbb{P}$-martingale almost surely.

Lemma 3.2 (Bayes' Formula). *If $\mathbb{Q} \sim \mathbb{P}$, then for all $t \geqslant s$ and any $\mathcal{F}_t$-measurable random variable $X \geqslant 0$, we have:*

$$\mathbb{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_s\right] = \frac{\mathbb{E}_{\mathbb{P}}\left[X Z_t \mid \mathcal{F}_s\right]}{Z_s}.$$

Proof. For all $t \geqslant s \geqslant 0$ and $A \in \mathcal{F}_s$, we have:

$$\mathbb{E}_{\mathbb{Q}}\left[X \mathbb{1}_A\right] = \mathbb{E}_{\mathbb{P}}\left[Z_\infty X \mathbb{1}_A\right] = \mathbb{E}_{\mathbb{P}}\left[Z_t X \mathbb{1}_A\right] = \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[Z_t X \mid \mathcal{F}_s\right] \mathbb{1}_A\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{\mathbb{E}_{\mathbb{P}}\left[Z_t X \mid \mathcal{F}_s\right]}{Z_s} \mathbb{1}_A\right],$$

which proves the claim. □

Recall that M is a martingale if for all $t > s$, we have $M_s = \mathbb{E}(M_t \mid \mathcal{F}_s)$.

Lemma 3.3 (Martingale and Change of Measure). *If $\mathbb{Q} \sim \mathbb{P}$, then:*

$$M \text{ is a } \mathbb{Q}\text{-martingale} \Longleftrightarrow MZ \text{ is a } \mathbb{P}\text{-martingale.}$$

The same statement holds for local martingales.

Proof. The equivalence directly follows from Bayes' formula. □

Doob's h-Transform

Let X be a diffusion defined by the SDE:

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t + \mu(X_t)\mathrm{d}t,$$

where B is a standard Brownian motion. Consider the probability $\mathbb{Q}$ defined by Doob's h-transform for the harmonic function $h(x)$ associated with the process $X(t)$, i.e.,

$$\mathbb{E}_x(h(X_t)) = h(x)$$

(or equivalently, $\mathcal{G}h = \frac{1}{2}\sigma^2 h'' + \mu h' = 0$, where $\mathcal{G}$ is the infinitesimal generator of the Markov process X).

We note that:

$$Z_t = h(X_t)$$

is a martingale. Indeed, for $s < t$, using the Markov property, we have:

$$\mathbb{E}_x(Z_t \mid \mathcal{F}_s) = \mathbb{E}_{X_s}(h(X_t)) = h(X_s) = Z_s.$$

We can define a probability such that:

$$\mathbb{Q}(X(t) \in S) = \mathbb{E}\left(\frac{h(X(t))}{h(X(0))} \mathbb{1}_{X(t) \in S}\right)$$

for a set $S \subset \mathbb{R}^2$.

Under $\mathbb{Q}$, the process $Z(t)$ is a Markov process with the transition kernel:

$$\widetilde{P}_t(x_0, x) = \frac{h(x)}{h(x_0)} P_t(x_0, x),$$

where P_t is the transition kernel of $X(t)$ under $\mathbb{P}$.

We now state the Girsanov theorem adapted to our context.

Theorem 3.4 (Girsanov's Theorem). *Let L_t be the stochastic logarithm of $Z_t = h(X_t)$, defined by:*

$$\mathrm{d}L_t = \frac{\mathrm{d}Z_t}{Z_t}.$$

Then:

$$\widetilde{B}_t = B_t - \langle B, L \rangle_t$$

is a Brownian motion under $\mathbb{Q}$.

Proof. We aim to show that $Z_t\widetilde{B}_t$ is a $\mathbb{P}$-martingale, which implies that $\widetilde{B}_t$ is a $\mathbb{Q}$-martingale by Lemma 3.3.

First, note that $Z_t \mathrm{d}\langle L, B\rangle_t = \mathrm{d}\langle Z, B\rangle_t = \mathrm{d}\langle Z, \widetilde{B}\rangle_t$. Using the stochastic integration by parts formula, we obtain:

$$\mathrm{d}(Z_t\widetilde{B}_t) = \widetilde{B}_t\mathrm{d}Z_t + Z_t\mathrm{d}\widetilde{B}_t + \mathrm{d}\langle Z_t, \widetilde{B}_t\rangle = \widetilde{B}_t\mathrm{d}Z_t + Z_t\mathrm{d}B_t - Z_t\mathrm{d}\langle B, L\rangle_t + \mathrm{d}\langle Z, \widetilde{B}\rangle_t,$$

where the last term equals 0. Hence, this is a $\mathbb{P}$-martingale. Therefore, $\widetilde{X}_t$ is a $\mathbb{Q}$-martingale, and since its quadratic variation is $\langle \widetilde{X}_t\rangle = t$, we deduce by Lévy's characterization theorem that $\widetilde{X}_t$ is a $\mathbb{Q}$-Brownian motion. □

The following proposition states that under $\mathbb{Q}$, the process $X(t)$ is conditioned (in the sense of Doob's h-transform) to drift in a new direction.

Proposition 3.5 (Doob's h-Transform of a Diffusion)**.** *Under $\mathbb{Q}$, the process X remains a diffusion with the same diffusion coefficient σ and a new drift given by:*

$$\mu + \sigma^2 \nabla \log h.$$

Under the new measure $\mathbb{Q}$, the process (X_t) satisfies the SDE:

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}\widetilde{B}_t + \mu(X_t)\mathrm{d}t + \sigma^2(X_t)\frac{h'(X_t)}{h(X_t)}\mathrm{d}t,$$

where $\widetilde{B}_t$ is a Brownian motion under $\mathbb{Q}$.

Proof. This is a classical result [5, VIII (3.9), p. 358]. The proof heavily relies on Girsanov's theorem.

To see this, let L_t be the stochastic logarithm of $Z_t = h(X_t)$, defined by:

$$\mathrm{d}L_t = \frac{\mathrm{d}Z_t}{Z_t}.$$

Using Girsanov's theorem:

$$\widetilde{B}_t = B_t - \langle B, L\rangle_t$$

is a Brownian motion under $\mathbb{Q}$. By Itô's formula:

$$\mathrm{d}Z_t = \mathrm{d}(h(X_t)) = h'(X_t)\mathrm{d}X_t + \frac{1}{2}h''(X_t)\mathrm{d}\langle X\rangle_t,$$

and thus $\mathrm{d}\langle B, Z\rangle_t = h'(X_t)\mathrm{d}\langle B, X\rangle_t = h'(X_t)\sigma^2\mathrm{d}t$, giving:

$$\mathrm{d}\langle B, L\rangle_t = \frac{h'(X_t)}{h(X_t)}\sigma^2\mathrm{d}t.$$

□

Scale Function

We introduce the concept of the scale function for a diffusion. Let τ_x denote the hitting time of level x for a process X:

$$\tau_x := \inf\{t \geqslant 0 \,|\, X_t = x\}.$$

For $x \in [a, b]$ (with $a < b$), the scale probability is defined as:

$$s_{ab}(x) := \mathbb{P}_x(\tau_b < \tau_a).$$

In general, it can be shown [5, (3.2) VII §3, p. 301] that there exists a strictly monotone, continuous function h, called the scale function of the diffusion X, such that:

$$\forall x \in [a, b], \quad s_{ab}(x) := \mathbb{P}_x(\tau_b < \tau_a) = \frac{h(x) - h(a)}{h(b) - h(a)}.$$

A scale function is defined up to additive and multiplicative constants. If h is harmonic for the diffusion X, then h is a scale function. Below, we calculate the scale functions for Brownian motion and the Bessel process.

Lemma 3.6 (Scale Function for Brownian Motion)**.** *The scale function for Brownian motion is given by $S(x) = x$, and we say that Brownian motion is in natural scale.*

Proof. Let $(B_t)_{t \geqslant 0}$ be a real Brownian motion, and let $a < b$ with $x \in [a, b]$. Define the stopping time:

$$\tau := \inf\{t \geqslant 0 \,|\, B_t = a \text{ or } B_t = b\}.$$

Since Brownian motion is a martingale, we have by Doob's stopping theorem:

$$\mathbb{E}_x [B_\tau] = \mathbb{E}_x [B_0] = x.$$

Additionally, we compute:

$$\begin{aligned} \mathbb{E}_x [B_\tau] &= \mathbb{E}_x [B_\tau \mathbb{1}_{\tau_b < \tau_a}] + \mathbb{E}_x [B_\tau \mathbb{1}_{\tau_b \geqslant \tau_a}] \\ &= b\mathbb{E}_x [\mathbb{1}_{\tau_b < \tau_a}] + a\mathbb{E}_x [\mathbb{1}_{\tau_b \geqslant \tau_a}] \\ &= b\mathbb{P}_x (\tau_b < \tau_a) + a (1 - \mathbb{P}_x (\tau_b < \tau_a)) \\ &= (b - a)\mathbb{P}_x (\tau_b < \tau_a) + a. \end{aligned}$$

Thus, we obtain:

$$s_{ab}(x) = \mathbb{P}_x (\tau_b < \tau_a) = \frac{x - a}{b - a}.$$

□

Lemma 3.7 (Scale Function for a 3D Bessel Process)**.** *The scale function for a three-dimensional Bessel process is given by $S(x) = -\frac{1}{x}$.*

Proof. The proof is similar. □

Diffusion Conditioned to Exit on One Side in $(0, L)$

For a diffusion X and a harmonic scale function h, recall that:

$$\mathbb{P}_{x_0}(\tau_0 > \tau_L) = \frac{h(x_0) - h(0)}{h(L) - h(0)}.$$

The new measure $\mathbb{Q}$, conditioning the diffusion to reach L before 0 starting from x_0, is defined as:

$$\mathbb{Q}(\cdot) = \mathbb{P}(\cdot \,|\, \tau_0 > \tau_L).$$

It is given by the Radon-Nikodym density:

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_\infty} = \frac{\mathbb{1}_{\tau_0 > \tau_L}}{\mathbb{P}(\tau_0 > \tau_L)} = \frac{h(X_{\tau_L \wedge \tau_0}) - h(0)}{h(x_0) - h(0)},$$

where we use the fact that:

$$h(X_{\tau_L \wedge \tau_0}) = h(L)\mathbb{1}_{\tau_0 > \tau_L} + h(0)(1 - \mathbb{1}_{\tau_0 > \tau_L}).$$

We can normalize the harmonic scale function to $\widetilde{h}(x) = \frac{h(x) - h(0)}{h(x_0) - h(0)}$, so:

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \mathbb{E}\left(\widetilde{h}(X_{\tau_L \wedge \tau_0}) \middle| \mathcal{F}_t\right) = \widetilde{h}(X_{t \wedge \tau_L \wedge \tau_0}).$$

Here, the density $\widetilde{h}(X_{t \wedge \tau_L \wedge \tau_0})$ is a martingale under $\mathbb{P}$ starting from 1.

By Proposition 3.5 on the properties of an h-transform, under the new measure $\mathbb{Q}$, the process (X_t) satisfies the following SDE:

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}\widetilde{B}_t + \mu(X_t)\mathrm{d}t + \sigma^2(X_t)\frac{\widetilde{h}'(X_t)}{\widetilde{h}(X_t)}\mathrm{d}t, \quad t \in [0, \tau_L],$$

where $\widetilde{B}_t$ is a Brownian motion under $\mathbb{Q}$.

4 Bessel Process

Definition 4.1 (Three-Dimensional Bessel Process)**.** The following three definitions are equivalent:

1. Let $(B_t, t \geqslant 0)$ be a Brownian motion in $\mathbb{R}^3$. The process $(\|B_t\|, t \geqslant 0)$ is a three-dimensional Bessel process.

2. Let $(B_t, t \geqslant 0)$ be a Brownian motion in $\mathbb{R}$. The solution to the stochastic differential equation $\mathrm{d}X_t = \mathrm{d}B_t + \frac{\mathrm{d}t}{X_t}$ is a three-dimensional Bessel process.

3. Let $(B_t, t \geqslant 0)$ be a real Brownian motion starting from $x_0 \in \mathbb{R}_+$ under the probability $\mathbb{P}$. Let $\tau_0 = \inf\{t \geqslant 0 : B_t = 0\}$. Define the probability:

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{X_{t\wedge\tau_0}}{x_0}.$$

   Under the probability $\mathbb{Q}$, the process B is a three-dimensional Bessel process starting from x_0. This is the h-process of Brownian motion killed at 0 for $h(x) = x$. It is also known as Brownian motion conditioned to remain positive in the sense of Doob's h-transform.

Proof. 1 ⇔ 2: This result can be found in Proposition 3.3, page 252 of Revuz and Yor [5]. We detail the proof here.

Let (R_t) be a three-dimensional Bessel process defined as $R_t = \|B_t\|$, where B_t is a Brownian motion in $\mathbb{R}^3$. Using Itô's lemma:

$$R_t = \sqrt{(B_t^1)^2 + (B_t^2)^2 + (B_t^3)^2}.$$

Applying Itô's formula to $f(x) = \sqrt{x_1^2 + x_2^2 + x_3^2}$, we have:

$$\mathrm{d}R_t = \nabla f(B_t) \cdot \mathrm{d}B_t + \frac{1}{2}\Delta f(B_t)\,\mathrm{d}t.$$

Computing the terms:

$$\nabla f(x) = \frac{1}{\|x\|}(x_1, x_2, x_3), \quad \Delta f(x) = \frac{2}{\|x\|}.$$

Substituting, we obtain the following SDE for R_t:

$$\mathrm{d}R_t = \mathrm{d}W_t + \frac{1}{R_t}\mathrm{d}t,$$

where W_t is a Brownian motion. By the local stochastic Cauchy-Lipschitz theorem, the uniqueness in law of the solution to such an SDE can be established. □

Proof. 2 ⇔ 3: Let $(X_t)_{t\geqslant 0}$ be a Brownian motion starting at $x_0 \in (0, L)$. We aim to condition X_t to reach L before 0, analyze the SDE satisfied by this process, and then let $L \to \infty$.

Recall that:

$$\mathbb{P}_{x_0}(\tau_0 > \tau_L) = \frac{h(x_0) - h(0)}{h(L) - h(0)} = \frac{x_0}{L},$$

where the harmonic function is $h(x) = x$.

The new measure $\mathbb{Q}$, conditioning the Brownian motion to reach L before 0 starting from x_0, is:

$$\mathbb{Q}(\cdot) = \mathbb{P}(\cdot \,|\, \tau_0 > \tau_L).$$

It is given by the Radon-Nikodym density:

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_\infty} = \frac{\mathbb{1}_{\tau_0 > \tau_L}}{\mathbb{P}(\tau_0 > \tau_L)} = \frac{X_{\tau_L \wedge \tau_0}}{x_0}.$$

Thus:

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \mathbb{E}\left(\frac{X_{\tau_L\wedge\tau_0}}{x_0}\,\middle|\,\mathcal{F}_t\right) = \frac{X_{t\wedge\tau_L\wedge\tau_0}}{x_0}.$$

By Proposition 3.5 on the properties of an h-transform, under the new measure $\mathbb{Q}$, the process (X_t) satisfies the following SDE:

$$\mathrm{d}X_t = \mathrm{d}\widetilde{X}_t + \frac{h'(X_t)}{h(X_t)}\mathrm{d}t = \mathrm{d}\widetilde{X}_t + \frac{1}{X_t}\mathrm{d}t, \quad t \in [0, \tau_L],$$

where $\widetilde{X}_t$ is a Brownian motion under $\mathbb{Q}$. The drift $\frac{1}{X_t}$ acts as a repulsive force, preventing the process from reaching 0.

When $L \to \infty$, the time τ_L tends to infinity, and the process becomes a Brownian motion on $\mathbb{R}_+$ conditioned to never reach 0. This limiting process is the three-dimensional Bessel process, which satisfies the SDE:

$$\mathrm{d}X_t = \mathrm{d}\widetilde{X}_t + \frac{1}{X_t}\mathrm{d}t, \quad t \geqslant 0,$$

where $\widetilde{X}_t$ is a Brownian motion under the probability $\mathbb{Q}$ defined by:

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{X_{t\wedge\tau_0}}{x_0}.$$

□

Remark 4.2 (Recurrence and Singularity of $\mathbb{Q}$)**.** Under $\mathbb{P}$, Brownian motion almost surely hits 0:

$$\mathbb{P}(T_0 < \infty) = 1.$$

However, under $\mathbb{Q}$, the process is defined to avoid 0 with probability 1:

$$\mathbb{Q}(T_0 < \infty) = 0.$$

Thus, the measure $\mathbb{Q}$ is said to be singular (or foreign) with respect to $\mathbb{P}$.

Remark 4.3 (Alternative Proof of 2 ⇔ 3)**.** See the article by Pinsky [] on diffusions conditioned to remain within a given domain. The summary below is from [1].

We aim to illustrate the method used by Pinsky to study a Brownian motion B on $\mathbb{R}^d$ conditioned to remain positive, i.e., in the domain $(0, \infty)$.

Let $\tau_0 = \inf\{t \geqslant 0 \,|\, X_t = 0\}$. Consider a Brownian motion starting at $x > 0$, $(B, \mathbb{P}_x)$, which satisfies the stochastic differential equation (SDE):

For a fixed time $T > 0$, define the measure:

$$\mathbb{Q}_x^T(\cdot) = \mathbb{P}_x(\cdot \,|\, \tau_D > T),$$

which corresponds to the process conditioned to remain positive up to time T. Let the function:

$$g^T(x) = \mathbb{P}_x(\tau_D > T), \quad x \in D.$$

Moreover:

$$\lim_{T\to\infty} \frac{\nabla g^T(x)}{g^T(x)} = \frac{\nabla\varphi_0(x)}{\varphi_0(x)},$$

uniformly over compact subsets of D, where φ_0 is the eigenfunction of the generator $\mathcal{L}$ with Dirichlet boundary conditions, associated with the smallest eigenvalue λ_0. It is assumed that $\varphi_0 > 0$ in D and $\varphi_0 = 0$ on the boundary ∂D.

The process defined by:

$$M_t^T = \frac{g^{T-t}(X_{t\wedge\tau_D})}{g^T(X_0)},$$

is a martingale under $\mathbb{P}_x$, and for any bounded, $\mathcal{F}_t$-measurable random variable H:

$$\mathbb{E}_{\mathbb{Q}_x^T}[H] = \mathbb{E}_x[HM_t^T].$$

Furthermore:

$$\mathbb{E}[M_t^T \mid \mathcal{F}_s] = M_s^T, \quad \text{for } t \geqslant s \geqslant 0.$$

Under the measure $\mathbb{Q}_x^T$, the process X never leaves the domain D. According to Doob's h-transform, the process X under $\mathbb{Q}_x^T$ follows an Itô diffusion with diffusion matrix $\sigma|_D$ and drift:

$$\vec{b}^T(x) = b(x) + \frac{\nabla g^T(x)}{g^T(x)}, \quad x \in D.$$

As $T \to \infty$, and under certain technical assumptions on the behavior of g^T, the process $(X, \mathbb{Q}_x^T)$ converges in law to a weak solution $(X, \mathbb{Q}_x)$ of the SDE with drift:

$$\widetilde{b}(x) = \lim_{T\to\infty} \vec{b}^T(x) = b(x) + (\sigma\sigma^T)(x)\frac{\nabla\varphi_0(x)}{\varphi_0(x)}, \quad x \in D.$$

The law $(X, \mathbb{Q}_x)$ can be interpreted as that of the original process conditioned never to leave D.

Proposition 4.4. *If R_t is a three-dimensional Bessel process, then R_t^{-1} is a local martingale.*

Proof. The result and its proof can be found in Proposition 3.3, page 252 of the book by Revuz and Yor [5]. Let $f(R_t) = \frac{1}{R_t}$. Applying Itô's formula:

$$\mathrm{d}f(R_t) = f'(R_t)\mathrm{d}R_t + \frac{1}{2}f''(R_t)(\mathrm{d}R_t)^2,$$

with

$$f'(x) = -\frac{1}{x^2} \quad \text{and} \quad f''(x) = \frac{2}{x^3}.$$

Substituting the SDE for R_t, $\mathrm{d}R_t = \mathrm{d}W_t + \frac{\mathrm{d}t}{R_t}$, and $(\mathrm{d}R_t)^2 = \mathrm{d}t$, we get:

$$\mathrm{d}f(R_t) = -\frac{1}{R_t^2}\left(\mathrm{d}W_t + \frac{\mathrm{d}t}{R_t}\right) + \frac{1}{R_t^3}\mathrm{d}t.$$

Simplifying:

$$\mathrm{d}f(R_t) = -\frac{\mathrm{d}W_t}{R_t^2}.$$

Thus, the drift term vanishes, and $f(R_t) = \frac{1}{R_t}$ is a local martingale. □

Proposition 4.5 (Transition Density of a Three-Dimensional Bessel Process)**.** *The transition density of the three-dimensional Bessel process is given by:*

$$p_t(a,b) = (a/t)(b/a)^{3/2} I_{1/2}(ab/t)\exp\left(-\left(a^2+b^2\right)/2t\right) \text{ for } a,b>0,$$

where $I_{1/2}$ is the modified Bessel function of the first kind of index $1/2$, and:

$$p_t(0,b) = \Gamma(3/2)2^{1/2}t^{-3/2}b^2\exp\left(-b^2/2t\right).$$

Proof. See Williams' book, p. 748, or Revuz and Yor, p. 251 [5]. □

5 Pitman's Representation Theorem for the 3D Bessel Process

Proposition 5.1 (Williams' Time Reversal)**.** *Let B be a Brownian motion starting at 0, and let R be a 3D Bessel process. Define $\gamma_a = \sup\{t \geqslant 0 : R_t = a\}$. Then:*

$$(R_u, u \leqslant \gamma_a) \overset{(d)}{=} (a - B_{T_a - u}, u \leqslant T_a)\,.$$

Proof. A key ingredient of this proof is the characterization 3 of Definition 4.1, which connects the Bessel process to Brownian motion via a Doob h-transform. The proof uses Nagasawa's theorem [2, Thm 3.3]. For details, see Exercise 3.3 of the book by Mallein and Yor [2], which corrects Corollary 3.4. This result is also stated and proven in Theorem (4.5) and Corollary (4.6), pages 315–317, of the book by Revuz and Yor [5]. Additionally, see Theorem 1.6.1 of the book by Yen and Yor [6]. □

The previous proposition is a key element in the proof of Pitman's theorem.

Theorem 5.2 (Pitman's Representation Theorem)**.** *Let W_t be a standard Brownian motion, and $I_t = \inf_{s \leqslant t} W_s$. Let $(R_t)_{t\geqslant 0}$ be a 3D Bessel process, and let $J_t = \inf_{s\geqslant t} R_s$ denote its future infimum. Then:*

$$(W_t - 2I_t, -I_t)_{t\geqslant 0} \overset{(law)}{=} (R_t, J_t)_{t\geqslant 0}\,.$$

Proof. There are multiple proofs of Pitman's theorem. For example, one can refer to the book by Williams (1974) (Thm 3.4, page 751), which uses transition probabilities. Another reference is the classic book by Revuz and Yor [5, Thm 3.5, page 253]. Below, we detail the proof given in Yen and Yor's book [6, Thm 3.1.1, page 33].

Levy's representation theorem implies that the following two statements are equivalent:

$$(W_t - 2I_t; t \geqslant 0) \overset{\text{(law)}}{=} (R_t; t \geqslant 0)\,,$$

and:

$$(|B_t| + L_t; t \geqslant 0) \overset{\text{(law)}}{=} (R_t; t \geqslant 0)\,.$$

By a scaling argument, it suffices to show that:

$$(|B_t| + L_t; t \leqslant \tau_1) \overset{\text{(law)}}{=} \left(R_t; t \leqslant \gamma^1\right), \tag{2}$$

where $\tau_1 = \inf\{t : L_t \geqslant 1\} = \sup\{t : |B_t| + L_t = 1\}$, and $\gamma^1 = \sup\{t : R_t = 1\}$.

By Proposition 1.9, we have:

$$(B_{\tau_1 - t}; t \leqslant \tau_1) \overset{\text{(law)}}{=} (B_t; t \leqslant \tau_1)\,,$$

which also holds jointly with:

$$(1 - L_{\tau_1 - t}; t \leqslant \tau_1) \overset{\text{(law)}}{=} (L_t; t \leqslant \tau_1)\,.$$

Thus, (2) is equivalent to:

$$(|B_{\tau_1 - t}| + (1 - L_{\tau_1 - t})\,; t \leqslant \tau_1) \overset{\text{(law)}}{=} \left(R_t; t \leqslant \gamma^1\right).$$

By Levy's representation theorem 2.2, this is equivalent to:

$$(1 - B_{T_1 - t}; t \leqslant T_1) \overset{\text{(law)}}{=} \left(R_t; t \leqslant \gamma^1\right),$$

which follows from Williams' time reversal (Proposition 5.1). □

Remark 5.3 (Alternative Formulation of Pitman's Theorem)**.** Let $S_t = \sup_{s \leqslant t}(W_s)$. Then:

$$(2S_t - W_t, S_t; t \geqslant 0) \overset{\text{(law)}}{=} (R_t, J_t; t \geqslant 0) \, .$$

Corollary 5.4 (Uniform Distribution Conditional on a Bessel Process)**.** *Let $R_t = W_t - 2I_t$, $\mathcal{R}_t = \sigma\{R_s; s \leqslant t\}$, and T be an $\mathcal{R}_t$-stopping time. Then, conditionally on $\mathcal{R}_T$, the random variable $-I_t$ (and consequently, $W_t - I_t$) is uniformly distributed over $[0, R_T]$. Specifically, for $0 < y < R_T$:*

$$\mathbb{P}(-I_t \leqslant y \,|\, R_T) = \mathbb{P}(W_t - I_t \leqslant y \,|\, R_T) = \frac{y}{R_T}.$$

Proof. Using Pitman's theorem, the statement of the corollary is equivalent to: If $(R^a_s; s \geqslant 0)$ is a $\mathrm{BES}_a(3)$ process, then $\inf_{s \geqslant 0} R^{(a)}_s$ is uniformly distributed over $[0, a]$.

From Proposition 4.4, we know that $1/R_t$ is a local martingale, and it converges to 0 as $t \to \infty$ (since a 3D Bessel process diverges to infinity). Thus, for $a > y$, by Doob's stopping theorem:

$$\frac{1}{a} = \mathbb{E}_a\left(\frac{1}{R_{T_y}}\right) = \frac{1}{y}\mathbb{P}(T_y < \infty) + 0 \cdot \mathbb{P}(T_y = \infty) = \frac{1}{y}\mathbb{P}(\inf R \leqslant y),$$

so:

$$\mathbb{P}(\inf R \leqslant y) = \frac{y}{a}.$$

□

Remark 5.5. The process I_t is not a Markov process. We have seen that for $a = 0, 1, 2$, the process $W_t - aI_t$ is a Markov process. These are the only values of a for which this property holds.

Remark 5.6 (Pitman Transformations)**.** Pitman's transformation is often defined as:

$$\mathcal{P}f(t) = f(t) - 2 \inf_{0 \leqslant s \leqslant t} f(s),$$

and sometimes as:

$$\widetilde{\mathcal{P}}f(t) = 2 \sup_{0 \leqslant s \leqslant t} f(s) - f(t).$$

Note that $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are not the same transformations!

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