Dao Van Thinh's Report on "Postes Rouges CNRS 2024"

1 Basic information

Visitor: Dr. Dao Van Thinh
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Dates of stay: from 24-8-2024 to 22/11/2024
The proposed research project: Differential Galois group and Tannakian group over a general base. The project aims to develop the structure theory of formal meromorphic connections over a ring, which generalizes the known Jordan-Levelt theorem over a field.

2 Achievements

Discussing with Prof. João Pedro dos Santos, we obtained the following results:

- The first discussed problem is to develop the structure theory of formal meromorphic connections over a ring, which generalizes the known Jordan-Levelt theorem over a field (see [ABC]). Since the formal meromorphic connections over rings are too general to obtain an "analog" Jordan-Levelt structure, we restrict to the subcategory of prudent connections and show that this is a Tannakian category. Notice that prudence only makes sense in case our ring is a complete local ring, so we may consider the case of complete discrete valuation ring (cDVR) for simplicity. With that consideration and restriction, we obtain a theorem describing the structure of formal prudent meromorphic connections over a cDVR. This work will become a paper soon.

- Another discussed problem is about Nori's conjecture on the base changed property of fundamental groups (see [Nor82]. Let's fix an algebraically closed field k of characteristic $p > 0$ and a smooth, projective, and connected curve C over k. We denote $Fr : C \to C$ the absolute Frobenius morphism and define

  $S(C, r, t) = \{$isomorphism classes of stable vector bundles of rank r over Cwhose pull-back by t

  Then Mehta and Subramanian in [MS02] showed that the Nori fundamental group of C has base-change property if and only if the numbers $|S(C, r, t)|$ are

finite for all $r, t > 0$. In the same paper, they provided an example of a curve not satisfying the above conjecture, but their counter-example is not smooth. Later Pauly [Pa07] constructed a curve C of genus 2 such that $S(C, 2, 4)$ is infinite. Continuing in this direction, dos Santos [dS12] constructed more examples of hyperelliptic curves over small characteristic. His method is to correspond the above set of Frobenius trivial stable vector bundles to a set of irreducible representations of an algebra. This algebra is called the p-curvature algebra, which comes from Cartier's theorem on p-curvature to pass from Frobenius trivial vector bundle to vector bundles with connections. Then Letzter's procedure [**?**, Le08]ill help us to detect the occurrence of an infinity of isomorphism classes of irreducible representations (with fixed rank) of that algebra.
Our observation was that we prove the infinity of isomorphism classes of irreducible representations (with fixed rank) of the $p-$curvature algebra of hyperelliptic curves in small genus and characteristic theoretically. This gave us a better view of Nori's conjecture and also helped us to construct a family of smooth curves not satisfy that conjecture. This work is a paper in progress.

- We attended Séminaire Méditerranéen de Géométrie Algébrique which was held in Genova on 10-11 September 2024.

- I delivered a seminar talk in University of Trieste, Italy on 12 November 2024, and here is the abstract:
Title: On the structure of formal connections over a complete discrete valuation ring.
Abstract: Abstract: Given a differential equation (DE) on a variety X, it is highly desired to compute or understand its Galois differential group. Over the complex field, this problem has been studied extensively by using complex analysis. We automatically face an algebraic problem when replacing the complex field with an arbitrary field k of characteristic 0. Thanks to Deligne who systematically treated the algebraic differential equations. This story went further under the work of Katz. Following Tannakian's (or Riemann-Hilbert correspondence) philosophy, starting with a differential equation, we want to find a representation of its Galois group (or a monodromy representation). To do so, we must define a fiber functor from the category of DEs to the category of vector spaces. The point is that: what if we don't have any point to take fiber? Here is an example. Let k be a field of characteristic 0, and $K = k((t))$ the field of Laurent series. Let X be the affine line over k punctured at 0, and write $D.E.(X/k)$ (and $D.E.(K/k)$) to be the category of differential equations on X/k (and K/k respectively). Then we have a natural inverse image functor from $D.E.(X/k)$ to $D.E.(K/k)$, and Katz showed that this is indeed an equivalence if we restrict the source to the full subcategory of "special" objects. As a corollary, we obtain a fiber functor from $D.E.(K/k)$ to the category of $k-$vector spaces. That result is mainly based on the work of Turrittin-Levelt on the structure of an arbitrary object in $D.E.(K/k)$. In this talk, I will report our attempt to extend the above story to the case that the base field k is replaced by a complete discrete valuation ring. This is a work in progress with Prof. Phung Ho Hai and Prof. dos Santos.

3 Acknowledgements

I would like to thank the French National Centre for Scientific Research (CNRS) and the Institut Montpelliérain Alexander Grothendieck, Université de Montpellier (IMAG), for their financial support and hospitality during his research stay at IMAG through the program Postes Rouges CNRS 2024.
I want to thank Prof. Joao Pedro P. dos Santos for his time, enthusiasm, and his patience on all of my questions and problems. From him, I learned a lot. I also would like to thank the secretaries in IMAG and CNRS for their hospitality and support.

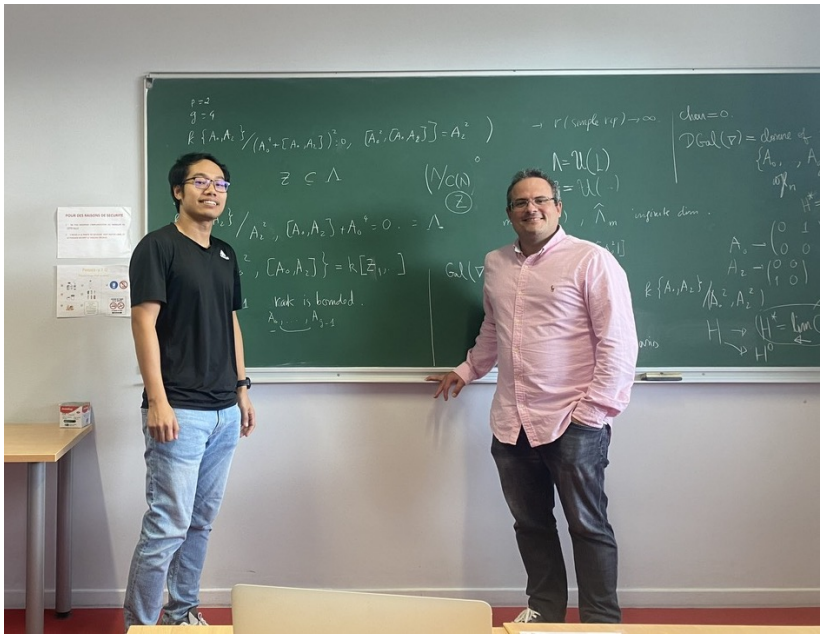

Figure 1: My host (right) and I were discussing in a room in the library of IMAG.

References

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